

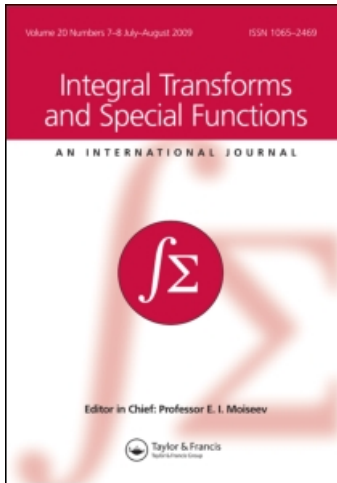
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On trigonometric series over integrals involving Bessel or Struve functions

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To find formulas for the summation of trigonometric series over integrals involving Bessel or Struve functions, we rely on trigonometric series involving Bessel or Struve functions, which are in turn obtained by using summation formulas for series over the product of two trigonometric functions. All these sums are expressed either as power series in terms of Riemann's ζ or Catalan's β function or Dirichlet functions η and λ , or, in certain cases, they are brought in so called closed form, which means that the infinite series are represented by finite sums. Important limiting values cases are considered too.

Keywords: Riemann's; Catalan's function; Bessel; Struve and Dirichlet functions

MSC (2000): Primary: 33C10; Secondary: 11M06; 65B10

1. Introduction

In this article, we deal with finding the sum of the series

$$I_{\alpha}^{D,f} = \sum_{n=1}^{\infty} \frac{(s)^{n-1} D_{\nu}((an - b)x)}{(an - b)^{\alpha}} f((an - b)z), \quad (1)$$

where $a = \left\{ \frac{1}{2} \right\}$, $b = \left\{ \frac{0}{1} \right\}$, $s = 1$ or -1 , $\alpha \in \mathbb{R}^+$, $f = \sin$ or $f = \cos$, and, $D_{\nu}(x)$ denotes an integral $B_{\nu, \phi}(x)$ or $S_{\nu, \phi}$, defined by

$$B_{\nu, \phi}(x) = \int_0^1 J_{\nu}(xy)\phi(y)dy, \quad S_{\nu, \phi}(x) = \int_0^1 \mathbf{H}_{\nu}(xy)\phi(y)dy, \quad (2)$$

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Table 1. Parameters and convergence regions.

a	b	s	c	F	for
1	0	1	1	ζ	$0 < x < 2\pi$
		-1	0	η	$-\pi < x < \pi$
2	1	1	$\frac{1}{2}$	λ	$0 < x < \pi$
		-1	0	β	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

where J_ν and \mathbf{H}_ν are Bessel or Struve functions of the first kind and order $\nu \in \mathbb{R}$. To obtain the sum of the series (1) we do not have to previously calculate integrals $D_\nu((an - b)x)$, which are not necessarily found elementarily. However, if we are able to do so, the series (1) will take a different form, leading possibly to a new class of summation formulas. At first, we assume that ϕ is integrable. Yet, in order to extend the class of summable series, we admit that ϕ is differentiable on $(0, 1)$, but not bounded in the neighbourhood of 0 or 1, or even not integrable on $(0, 1)$, however, such that there exists at least one of the integrals (2). We further require that the functions $y^k \phi(y)$ ($k \in \mathbb{N}$) are integrable on $(0, 1)$ as well.

Obtaining the sum of the series (1) relies on the summation of some trigonometric series (see [6]) in terms of Riemann’s ζ or Catalan’s β function or Dirichlet functions η and λ , in the form of a single formula, i.e.

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)x)}{(an - b)^\alpha} = \frac{c\pi}{2\Gamma(\alpha) f(\pi\alpha/2)} x^{\alpha-1} + \sum_{i=0}^{\infty} \frac{(-1)^i F(\alpha - 2i - \delta)}{(2i + \delta)!} x^{2i+\delta}, \quad (3)$$

where $\alpha > 0$, $a = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$, $b = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$, $s = 1$ or -1 , and $f = \begin{Bmatrix} \sin \\ \cos \end{Bmatrix}$, $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$. The values for F and c are in the Table 1, where ζ is Riemann’s zeta function $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$, η and λ are Dirichlet functions $\eta(z) = \sum_{k=1}^{\infty} (-1)^{k-1} k^{-z} = (1 - 2^{1-z})\zeta(z)$, $\lambda(z) = \sum_{k=0}^{\infty} (2k + 1)^{-z} = (1 - 2^{-z})\zeta(z)$ and β is Catalan’s function $\beta(z) = \sum_{k=0}^{\infty} (-1)^k (2k + 1)^{-z}$.

We note that the functions ζ, η, λ are analytic in the whole complex plane except for $z = 1$, where they have a pole. The integral representation $\beta(z) = 1/\Gamma(z) \int_0^\infty (x^{z-1} e^x)/(e^{2x} + 1) dx$ of Catalan’s function defines an analytical function for $\Re z \geq 1$; but, also it satisfies the functional equation $\beta(z) = (\pi/2)^{z-1} \Gamma(1 - z) \cos(\pi z/2) \beta(1 - z)$ extending beta to the left side of the complex plane $\Re z < 1$.

2. Preliminaries

We place the integral $D(x)$ in Equation (1), and check whether the interchange of summation and integration

$$\begin{aligned} I_\alpha^{D,f} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)z)}{(an - b)^\alpha} \int_0^1 \varphi_\nu((an - b)xy) \phi(y) dy \\ &= \int_0^1 \left(\sum_{n=1}^{\infty} \frac{(s)^{n-1} \varphi_\nu((an - b)xy)}{(an - b)^\alpha} f((an - b)z) \right) \phi(y) dy \end{aligned} \quad (4)$$

may take place. Here is $\alpha > 0, a = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} b = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, s = 1$ or $-1, f = \sin$ or \cos , and φ_ν is a Bessel J_ν or Struve function \mathbf{H}_ν . In order to justify Equation (4), we prove uniform convergence of the series

$$S_\alpha^{\varphi, f} = \sum_{n=1}^\infty \frac{(s)^{n-1} \varphi_\nu((an - b)xy) f((an - b)z)}{(an - b)^\alpha} \tag{5}$$

with respect to $y \in (0, 1)$. For this purpose, we use an integral representation of the Bessel or Struve function [1]

$$\varphi_\nu(t) = \frac{2(t/2)^\nu}{\Gamma(1/2)\Gamma(\nu + 1/2)} \int_0^{\pi/2} \sin^{2\nu} \theta g(t \cos \theta) d\theta, \tag{6}$$

where $\nu > -1/2, \varphi_\nu = \begin{Bmatrix} J_\nu \\ \mathbf{H}_\nu \end{Bmatrix} g = \begin{Bmatrix} \cos \\ \sin \end{Bmatrix}$. After replacing φ_ν in Equation (5) with the right-hand side integral of Equation (6), we shall first prove that we may interchange integration and summation, i.e.

$$\begin{aligned} S_\alpha^{\varphi, f} &= \frac{2(xy/2)^\nu}{\sqrt{\pi}\Gamma(\nu + (1/2))} \sum_{n=1}^\infty \frac{(s)^{n-1} f((an - b)z)}{(an - b)^{\alpha-\nu}} \int_0^{\pi/2} \sin^{2\nu} \theta g((an - b)xy \cos \theta) d\theta \\ &= \frac{2(xy/2)^\nu}{\sqrt{\pi}\Gamma(\nu + (1/2))} \int_0^{\pi/2} \sin^{2\nu} \theta \left(\sum_{n=1}^\infty \frac{(s)^{n-1} f((an - b)z)}{(an - b)^{\alpha-\nu}} g((an - b)xy \cos \theta) \right) d\theta, \end{aligned} \tag{7}$$

by showing uniform convergence of the right-hand series with respect to $(y, \theta) \in (0, 1) \times [0, \pi/2]$ on the basis of Dirichlet's test, which says that (see [3]) the series $\sum_{n=0}^\infty a_n(y)b_n(y)$ is uniformly convergent in D , if the partial sums of $\sum_{n=0}^\infty a_n(y)$ are uniformly bounded in D and the sequence $b_n(y)$, being monotonic for every fixed y , uniformly converges to 0.

LEMMA 1 *The series (5) converges uniformly with respect to y on $(0, 1)$.*

Proof In order to prove this, we treat the right-hand series of Equation (7) as a function of $y \in (0, 1)$, regarding x, z and θ as variable parameters. By making use of an elementary trigonometric identity, the product of f and g is represented as a sum of two trigonometric functions \sin or \cos . Consequently, the series in question is split up into two series of the type

$$\sum_{n=1}^\infty \frac{(s)^{n-1} \tau((an - b)(z \pm xy \cos \theta))}{(an - b)^{\alpha-\nu}}, \tag{8}$$

where $\tau = \sin$ or $\tau = \cos$. Let us suppose first that $a = 1, b = 0$. If $s = 1$, then there holds

$$\left| \sum_{k=1}^n \tau(k(z \pm xy \cos \theta)) \right| \leq \frac{1}{\sin(z \pm xy \cos \theta)/2} \leq \frac{1}{\sin \varepsilon}$$

for $0 < z \pm xy \cos \theta < 2\pi$, because $\sin(z \pm xy \cos \theta)/2 \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \leq (z \pm xy \cos \theta)/2 \leq \pi - \varepsilon$. Similarly, if $s = -1$, we would have

$$\left| \sum_{k=1}^n (-1)^{k-1} \tau(k(z \pm xy \cos \theta)) \right| \leq \frac{1}{\cos(z \pm xy \cos \theta)/2} \leq \frac{1}{\sin \varepsilon}$$

for $-\pi < z \pm xy \cos \theta < \pi$, because $\cos(z \pm xy \cos \theta)/2 \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-\pi/2 + \varepsilon \leq (z \pm xy \cos \theta)/2 \leq \pi/2 - \varepsilon$.

We now suppose $a = 2, b = 1$. If $s = 1$, then

$$\left| \sum_{k=1}^n \tau((2k - 1)(z \pm xy \cos \theta)) \right| \leq \frac{1}{\sin(z \pm xy \cos \theta)} \leq \frac{1}{\sin \varepsilon}$$

for $0 < z \pm xy \cos \theta < \pi$, because $\sin(z \pm xy \cos \theta) \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \leq z \pm xy \cos \theta \leq \pi - \varepsilon$. Finally if $s = -1$, then

$$\left| \sum_{k=1}^n (-1)^{k-1} \tau((2k - 1)(z \pm xy \cos \theta)) \right| \leq \frac{1}{\cos(z \pm xy \cos \theta)} \leq \frac{1}{\sin \varepsilon}$$

for $-\pi/2 < z \pm xy \cos \theta < \pi/2$, because $\cos(z \pm xy \cos \theta) \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-\pi/2 + \varepsilon \leq z \pm xy \cos \theta \leq \pi/2 - \varepsilon$.

On the one hand, all the partial sums are uniformly bounded with respect to $(y, \theta) \in [0, 1] \times [0, \pi/2]$, and considering that $0 \leq y \cos \theta \leq 1$, we are able to determine boundaries for x and z , i.e. the convergence regions for the series (1). They are in Table 2. For instance, in the first case, from $0 \leq y \cos \theta \leq 1$, we have immediately $-|x| \leq \pm xy \cos \theta \leq |x|$, and to come to the condition $0 < z \pm xy \cos \theta < 2\pi$, it is necessary to take $|x| < z < 2\pi - |x|$; hence, there follows $|x| < \pi$. In a similar way the rest of the convergence regions are determined. Note that the boundaries for x are the same as for $xy \cos \theta$, because $0 \leq y \cos \theta \leq 1$.

On the other hand, it is obvious that in each case the sequence $1/(an - b)^{\alpha-\nu}$ monotonically tends to 0 for $\alpha > \nu$, which proves uniform convergence of the right-hand series in Equation (7) with respect to $(y, \theta) \in (0, 1) \times [0, \pi/2]$. So the interchange of integration and summation in Equation (7) is permitted. There still remains to prove that the left-hand series in Equation (7) uniformly converges with respect to $y \in (0, 1)$. Namely, relying on Equation (7) and uniform convergence of the right-hand series in Equation (7) with respect to $(y, \theta) \in (0, 1) \times [0, \pi/2]$, we state that for an arbitrary $\varepsilon > 0$, there exists k_0 , so that $k \geq k_0$ implies

$$\begin{aligned} & \left| \frac{2\Gamma(\nu + 1)}{\sqrt{\pi}\Gamma(\nu + (1/2))} \sum_{n=k+1}^{\infty} \frac{(s)^{n-1} f((an - b)z)}{(an - b)^{\alpha-\nu}} \int_0^{\pi/2} \sin^{2\nu} \theta g((an - b)xy \cos \theta) d\theta \right| \\ &= \left| \int_0^{\pi/2} \sin^{2\nu} \theta \left(\frac{2\Gamma(\nu + 1)}{\sqrt{\pi}\Gamma(\nu + (1/2))} \sum_{n=k+1}^{\infty} \frac{(s)^{n-1} f((an - b)z)}{(an - b)^{\alpha-\nu}} g((an - b)xy \cos \theta) \right) d\theta \right| \\ &\leq \int_0^{\pi/2} \sin^{2\nu} \theta \left| \frac{2\Gamma(\nu + 1)}{\sqrt{\pi}\Gamma(\nu + (1/2))} \sum_{n=k+1}^{\infty} \frac{(s)^{n-1} f((an - b)z)}{(an - b)^{\alpha-\nu}} g((an - b)xy \cos \theta) \right| d\theta < \varepsilon. \end{aligned} \tag{9}$$

Table 2. Parameters and convergence regions.

a	b	s	c	F	Convergence region
1	0	1	1	ζ	$K_1 = \{(x, z) \mid -\pi < x < \pi, x < z < 2\pi - x \}$
1	0	-1	0	η	$K_2 = \{(x, z) \mid -\pi < x < \pi, x - \pi < z < \pi - x \}$
2	1	1	$\frac{1}{2}$	λ	$K_3 = \{(x, z) \mid -\frac{\pi}{2} < x < \frac{\pi}{2}, x < z < \pi - x \}$
2	1	-1	0	β	$K_4 = \{(x, z) \mid -\frac{\pi}{2} < x < \frac{\pi}{2}, x - \frac{\pi}{2} < z < \frac{\pi}{2} - x \}$

Hence, we conclude that the convergence speed of the first series in Equation (9) does not depend on y , meaning that the left-hand side series in Equation (7) uniformly converges with respect to $y \in (0, 1)$, and so does the series (5). ■

LEMMA 2 *Let $\phi(y)$ be integrable. Then there holds Equation (4).*

Proof Let us denote by

$$S_{\alpha,k}^{\phi,f} = \sum_{n=1}^k \frac{(s)^{k-1} \varphi_v((an - b)xy) f((an - b)z)}{(an - b)^\alpha} \tag{10}$$

the k th partial sum of the series (5). We have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)z)}{(an - b)^\alpha} \int_0^1 \varphi_v((an - b)xy) \phi(y) dy \\ &= \lim_{k \rightarrow \infty} \sum_{n=1}^k \frac{(s)^{n-1} f((an - b)z)}{(an - b)^\alpha} \int_0^1 \varphi_v((an - b)xy) \phi(y) dy \\ &= \lim_{k \rightarrow \infty} \int_0^1 \left(\sum_{n=1}^k \frac{(s)^{n-1} \varphi_v((an - b)xy) f((an - b)z)}{(an - b)^\alpha} \right) \phi(y) dy \\ &= \int_0^1 \left(\sum_{n=1}^{\infty} \frac{(s)^{n-1} \varphi_v((an - b)xy) f((an - b)z)}{(an - b)^\alpha} \right) \phi(y) dy. \end{aligned}$$

The last passage is permitted because both the sequence (10) uniformly converges with respect to y on $(0, 1)$ (Lemma 1) and $\int_0^1 \phi(y) dy$ exists. ■

LEMMA 3 *Suppose that, for a differentiable on $(0, 1)$ and unbounded in the neighbourhood of 0 or 1 function ϕ , the integral $\int_0^1 \phi(y) dy$ does not converge. Let there exist at least one of the integrals $D((an - b)x)$ (D is B_ϕ or S_ϕ defined by Equation (2)), so that $|D((an - b)x)| \leq M_n(x)$, and for each corresponding x from Table 2, the sequence $M_n(x)/(an - b)^\alpha$, $\alpha > 0$, monotonically tends to 0. Then there holds Equation (4).*

Proof Because of the assumption that ϕ is a differentiable function, we know that it is continuous on each closed interval within $(0, 1)$, and, as it is not bounded in the neighbourhood of 0 or 1, without loss of generality, we can consider $[\delta, 1 - \delta]$, $0 < \delta < 1$. Continuous function on a closed interval is bounded, so referring again to Dirichlet’s test, we prove uniform convergence of the series (5) with respect to y , but this time on $[\delta, 1 - \delta]$, so that there holds

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)z)}{(an - b)^\alpha} \int_\delta^{1-\delta} \varphi_v((an - b)xy) \phi(y) dy \\ &= \int_\delta^{1-\delta} \left(\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)z) \varphi_v((an - b)xy)}{(an - b)^\alpha} \right) \phi(y) dy \quad (\alpha > 0). \tag{11} \end{aligned}$$

We regard the left-hand side series as a function of δ with variable parameters z and x . In view of the conditions, by virtue of Dirichlet’s test, the left-hand side series in Equation (11) converges

uniformly with respect to δ on $(0, 1)$. Hence, we have

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)z)}{(an - b)^\alpha} \int_{\delta}^{1-\delta} \varphi_\nu((an - b)xy) \phi(y) dy \\ = \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)z)}{(an - b)^\alpha} \int_0^1 \varphi_\nu((an - b)xy) \phi(y) dy, \end{aligned}$$

meaning that the right-hand side integral in Equation (11) converges, so there holds Equation (4). ■

3. Sum of the series over the product of a Bessel and a trigonometric function

The series involving the product of sine or cosine play a key role in finding the summation formula for the series (1). That is why we investigated them thoroughly in [5]. After representing the product of g and f as the sum of two trigonometric functions \sin or \cos , and applying Equation (3) to both of series in each of the particular cases, in [5] we obtained a general formula

$$\begin{aligned} T_\alpha^{f,g} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} g((an - b)xy \cos \theta) f((an - b)z)}{(an - b)^{\alpha-\nu}} \\ &= \frac{c\pi(-1)^{\delta(\delta-d)}}{4\Gamma(\alpha - \nu)h(\pi(\alpha - \nu)/2)} ((z + xy \cos \theta)^{\alpha-\nu-1} + (-1)^\delta (z - xy \cos \theta)^{\alpha-\nu-1}) \quad (12) \\ &\quad + \sum_{i=0}^{\infty} \frac{(-1)^{\delta(\delta-d)+i} F(\alpha - \nu - 2i - d)}{(2i + d)!} \sum_{j=0}^i \binom{2i + d}{2j + \delta} z^{2i-2j+d-\delta} (xy \cos \theta)^{2j+\delta}, \end{aligned}$$

where $g = \begin{Bmatrix} \sin \\ \cos \end{Bmatrix}$, $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ and $d = \begin{Bmatrix} 0 & f=g \\ 1 & f \neq g \end{Bmatrix}$, $h = \begin{Bmatrix} \cos & f=g \\ \sin & f \neq g \end{Bmatrix}$. All the other relevant parameters are in Table 2.

When on the right-hand side of Equation (12) appears $h = \sin$ and $\alpha - \nu = 2m$ or $h = \cos$ and $\alpha - \nu = 2m - 1$, where $m \in \mathbb{N}$, one should take limit. However, if $\alpha - \nu - d = 2m$ and $F = \zeta, \eta, \lambda$ or $\alpha - \nu - d = 2m - 1$ and $F = \beta$ ($m \in \mathbb{N}$), the sum of the series on the right-hand side of Equation (12) consists of a finite number of terms because of the vanishing functions ζ, η, λ at even negative integers, and the function β at odd negative integers. So, for this choice of parameters, the formula (12) is brought into so called closed form (see [5])

$$\begin{aligned} T_{2m+d+\varepsilon}^{f,g} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} g((an - b)xy \cos \theta) f((an - b)z)}{(an - b)^{2m+d+\varepsilon}} \\ &= \frac{c\pi}{2} \sum_{j=0}^m \binom{2m + d + \varepsilon - 1}{2j + \delta} \frac{(-1)^{\delta(\delta-d)} z^{2m-2j+d-\delta+\varepsilon-1} (xy \cos \theta)^{2j+\delta}}{(2m + d + \varepsilon - 1)! h(m\pi + (d + \varepsilon)\pi/2)} \quad (13) \\ &\quad + \sum_{i=0}^m \frac{(-1)^{\delta(\delta-d)+i} F(2m - 2i + \varepsilon)}{(2i + d)!} \sum_{j=0}^i \binom{2i + d}{2j + \delta} z^{2i-2j+d-\delta} (xy \cos \theta)^{2j+\delta}, \end{aligned}$$

where $g = \begin{Bmatrix} \sin \\ \cos \end{Bmatrix}$, $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, and independently of that $d = \begin{Bmatrix} 0 & f=g \\ 1 & f \neq g \end{Bmatrix}$, $h = \begin{Bmatrix} \cos & f=g \\ \sin & f \neq g \end{Bmatrix}$. For $F = \zeta, \eta, \lambda$, there holds $\varepsilon = 0$, but for $F = \beta$ it is $\varepsilon = 1$. The other relevant parameters are in Table 2. The

formula (13) comprises some particular results from [4], but it is more suitable for immediate obtaining sums of some infinite series as well.

Now, we find the sum (5) by virtue of the summation formula (12), replacing it in Equation (7) and developing the binomials $(z \pm xy \cos \theta)^{\alpha-\nu-1}$ into binomial series. After a rearrangement we obtain

$$S_{\alpha}^{\varphi, f} = \frac{(xy/2)^{\nu}}{G_{\nu}} \left(\frac{c\pi(-1)^{\delta(\delta-d)}}{2\Gamma(\alpha-\nu)h(\pi(\alpha-\nu)/2)} \sum_{j=0}^{\infty} \binom{\alpha-\nu-1}{2j+\delta} z^{\alpha-\nu-1-2j-\delta} (xy)^{2j+\delta} I_{2\nu, 2j+\delta} \right. \\ \left. + \sum_{i=0}^{\infty} \frac{(-1)^{\delta(\delta-d)+i} F(\alpha-\nu-2i-d)}{(2i+d)!} \sum_{j=0}^i \binom{2i+d}{2j+\delta} z^{2i-2j+d-\delta} (xy)^{2j+\delta} I_{2\nu, 2j+\delta} \right), \tag{14}$$

where, for the sake of simplicity, $G_{\nu} = \sqrt{\pi}\Gamma(\nu+1/2)$ and $I_{2\nu, 2j+\delta} = \int_0^{\pi/2} \sin^{2\nu} \theta \cos^{2j+\delta} \theta d\theta$. Introducing $\sin \theta = t$ in the last integral, we have

$$I_{2\nu, 2j+\delta} = \frac{1}{2} \int_0^1 (t^2)^{(2\nu+1)/2-1} (1-t^2)^{(2j+\delta+1)/2-1} d(t^2) = \frac{1}{2} B\left(\nu + \frac{1}{2}, j + \frac{\delta+1}{2}\right), \tag{15}$$

and come to the required summation formula for the series (5)

$$S_{\alpha}^{\varphi, f} = \frac{(-1)^{\delta(\delta-d)} c \sqrt{\pi} (xy/2)^{\nu}}{2\Gamma(\alpha-\nu)h(\pi(\alpha-\nu)/2)} \sum_{j=0}^{\infty} \binom{\alpha-\nu-1}{2j+\delta} z^{\alpha-\nu-2j-1-\delta} (xy)^{2j+\delta} G_j \\ + \frac{(xy/2)^{\nu}}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^{\delta(\delta-d)+i} F(\alpha-\nu-2i-d)}{(2i+d)!} \sum_{j=0}^i \binom{2i+d}{2j+\delta} z^{2i-2j+d-\delta} (xy)^{2j+\delta} G_j, \tag{16}$$

where $\alpha > \nu > -1/2$, $\varphi_{\nu} = \left\{ \begin{smallmatrix} J_{\nu} \\ \mathbf{H}_{\nu} \end{smallmatrix} \right\}$, $g = \left\{ \begin{smallmatrix} \cos \\ \sin \end{smallmatrix} \right\}$, $\delta = \left\{ \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right\}$; independently of that $d = \left\{ \begin{smallmatrix} 0 & f=g \\ 1 & f \neq g \end{smallmatrix} \right\}$ and $h = \left\{ \begin{smallmatrix} \cos & f=g \\ \sin & f \neq g \end{smallmatrix} \right\}$. The other relevant parameters are given in Table 2, and for the sake of brevity, we have introduced $G_j = \Gamma(j + (\delta + 1)/2) / \Gamma(j + \nu + 1 + \delta/2)$.

3.1. Limiting values

We shall now consider some important particular cases of the formula (16). If $h = \sin$ and $\alpha - \nu = 2m$ or $h = \cos$ and $\alpha - \nu = 2m - 1$, $m \in \mathbb{N}$ division by zero is not defined, and we have to take limit. After choosing $a = 1, b = 0, s = 1$, there must be $c = 1, F = \zeta$. Also if we set $\varphi_{\nu} = J_{\nu}, g = \cos, \delta = 0, f = \cos$, then $d = 0, h = \cos$, and we have

$$S_{\alpha}^{J, \cos} = \frac{\sqrt{\pi} (xy/2)^{\nu} z^{\alpha-\nu-1}}{2\Gamma(\alpha-\nu) \cos(\pi(\alpha-\nu)/2)} \sum_{j=0}^{\infty} \binom{\alpha-\nu-1}{2j} \frac{(xy/z)^{2j} \Gamma(j+1/2)}{\Gamma(j+\nu+1)} \\ + \frac{(xy/2)^{\nu}}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-z^2)^i \zeta(\alpha-\nu-2i)}{(2i)!} \sum_{j=0}^i \frac{(xy/z)^{2j} \binom{2i}{2j} \Gamma(j+1/2)}{\Gamma(j+\nu+1)}.$$

In order to take limit, we first denote $\sigma = \alpha - \nu$ and replace α with $\sigma + \nu$, then we find

$$\begin{aligned} \Phi_{2m-1,\nu}(x, y, z) &= \lim_{\sigma \rightarrow 2m-1} \left[\frac{\sqrt{\pi}(xy/2)^\nu z^{\sigma-1}}{2\Gamma(\sigma) \cos(\pi\sigma/2)} \sum_{j=0}^{m-1} \binom{\sigma-1}{2j} \frac{(xy/z)^{2j} \Gamma(j+1/2)}{\Gamma(j+\nu+1)} \right. \\ &\quad \left. + \frac{(xy/2)^\nu}{\sqrt{\pi}} \sum_{i=0}^{m-1} \frac{(-z^2)^i \zeta(\sigma-2i)}{(2i)!} \sum_{j=0}^i \frac{(xy/z)^{2j} \binom{2i}{2j} \Gamma(j+1/2)}{\Gamma(j+\nu+1)} \right] \\ &= \frac{(-z^2)^{m-1} (xy/2)^\nu}{(2m-2)! \sqrt{\pi}} \sum_{k=0}^{m-1} \frac{(xy/z)^{2k} \binom{2m-2}{2k} (\psi(2m-2k-1) + \gamma - \log z) \Gamma(k+1/2)}{\Gamma(\nu+k+1)} \\ &\quad + \frac{(xy/2)^\nu}{\sqrt{\pi}} \sum_{k=0}^{m-2} \frac{(-z^2)^k \zeta(2m-2k-1)}{(2k)!} \sum_{j=0}^k \frac{(xy/z)^{2j} \binom{2k}{2j} \Gamma(j+1/2)}{\Gamma(j+\nu+1)}, \end{aligned}$$

and

$$\begin{aligned} R_{2m-1,\nu}(x, y, z) &= \lim_{\sigma \rightarrow 2m-1} \frac{\sqrt{\pi}(xy/2)^\nu z^{\sigma-1}}{2\Gamma(\sigma) \cos(\pi\sigma/2)} \sum_{j=m}^{\infty} \binom{\sigma-1}{2j} \frac{(xy/z)^{2j} \Gamma(j+1/2)}{\Gamma(j+\nu+1)} \\ &= -\frac{(-z^2)^{m-1} (xy/2)^\nu}{(2m-1)! \sqrt{\pi}} \sum_{j=m}^{\infty} \frac{(xy/z)^{2j} \Gamma(j+1/2)}{\binom{2j}{2m-1} \Gamma(\nu+j+1)}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} S_{2m-1+\nu}^{J,\cos} &= \sum_{n=1}^{\infty} \frac{J_\nu(nxy)}{n^{2m-1+\nu}} \cos nz = \Phi_{2m-1,\nu}(x, y, z) + R_{2m-1,\nu}(x, y, z) \\ &\quad + \frac{(xy/2)^\nu}{\sqrt{\pi}} \sum_{i=m}^{\infty} \frac{(-z^2)^i \zeta(2m-1-2i)}{(2i)!} \sum_{j=0}^i \frac{(xy/z)^{2j} \binom{2i}{2j} \Gamma(j+1/2)}{\Gamma(j+\nu+1)}, \end{aligned}$$

which holds for $(x, z) \in K_1$ (see Table 2 on the page 4).

Example 1 Let $m = 3$. Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{J_\nu(nxy)}{n^{7+\nu}} \cos nz &= \frac{(xy/2)^\nu}{\sqrt{\pi}} \left(-\frac{49z^6}{14400\Gamma(\nu+1)} - \frac{25(xy)^2 z^4}{1152\Gamma(\nu+2)} - \frac{3(xy)^4 z^2}{128\Gamma(\nu+3)} \right. \\ &\quad \left. + \left(\frac{z^6}{720\Gamma(\nu+1)} + \frac{(xy)^2 z^4}{96\Gamma(\nu+2)} + \frac{(xy)^4 z^2}{64\Gamma(\nu+3)} + \frac{(xy)^6}{384\Gamma(\nu+4)} \right) \log z \right. \\ &\quad \left. + \left(\frac{z^4}{24\Gamma(\nu+1)} + \frac{(xy)^2 z^2}{8\Gamma(\nu+2)} + \frac{(xy)^4}{32\Gamma(\nu+3)} \right) \zeta(3) - \left(\frac{z^2}{2\Gamma(\nu+1)} \right. \right. \\ &\quad \left. \left. + \frac{(xy)^2}{4\Gamma(\nu+2)} \right) \zeta(5) + \frac{\zeta(7)}{\Gamma(\nu+1)} + \frac{z^6}{7!} \sum_{j=4}^{\infty} \frac{(xy/z)^{2j} \Gamma(j+1/2)}{\binom{2j}{7} \Gamma(\nu+j+1)} \right. \\ &\quad \left. + \sum_{i=4}^{\infty} \frac{(-1)^i \zeta(7-2i)}{(2i)!} \sum_{j=0}^i \frac{(xy)^{2j} z^{2i-2j} \binom{2i}{2j} \Gamma(j+1/2)}{\Gamma(j+\nu+1)} \right). \end{aligned}$$

3.2. Closed form cases

The summation formula (16) takes a closed form in certain cases. Namely, the series on the right-hand side truncates because of the vanishing of F functions, i.e. $\alpha - \nu - d = 2m$ if $F = \zeta, \eta, \lambda$ and $\alpha - \nu - d = 2m - 1$ if $F = \beta$ ($m \in \mathbb{N}$), so that we write $\alpha = \nu + 2m + d - \varepsilon$, and obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(s)^{n-1} \varphi_{\nu}((an - b)xy) f((an - b)z)}{(an - b)^{\nu+2m+d-\varepsilon}} \\ &= \frac{(-1)^{\delta(\delta-d)} c \sqrt{\pi} (xy/2)^{\nu}}{2\Gamma(2m + d - \varepsilon) h(m\pi + \pi(d - \varepsilon)/2)} \\ & \times \sum_{j=0}^m \binom{2m + d - \varepsilon - 1}{2j + \delta} z^{2m+d-\varepsilon-2j-\delta-1} (xy)^{2j+\delta} G_j \\ & + \frac{(xy/2)^{\nu}}{\sqrt{\pi}} \sum_{i=0}^m \frac{(-1)^{\delta(\delta-d)+i} F(2m - 2i - \varepsilon)}{(2i + d)!} \\ & \times \sum_{j=0}^i \binom{2i + d}{2j + \delta} z^{2i-2j+d-\delta} (xy)^{2j+\delta} G_j, \end{aligned} \tag{17}$$

where $\varphi_{\nu} = \left\{ \begin{smallmatrix} J_{\nu} \\ \mathbf{H}_{\nu} \end{smallmatrix} \right\}$, $g = \left\{ \begin{smallmatrix} \cos \\ \sin \end{smallmatrix} \right\}$, $\delta = \left\{ \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right\}$, $d = \left\{ \begin{smallmatrix} 0 & f=g \\ 1 & f \neq g \end{smallmatrix} \right\}$, $h = \left\{ \begin{smallmatrix} \cos & f=g \\ \sin & f \neq g \end{smallmatrix} \right\}$, $\varepsilon = 0$ if $F = \zeta, \eta, \lambda$ and $\varepsilon = 1$ if $F = \beta$. The parameters a, b, s, c, F and convergence regions are read from Table 2. Particular closed form cases found in the literature (see Example 6), can be obtained from Equation (17).

Example 2 Consider the formula (74.1.19) in [2]

$$\sum_{n=1}^{\infty} \frac{J_{\nu}(nx)}{n^{\nu+2}} \cos nx = \frac{1}{3\Gamma(\nu + 2)} 2^{-\nu-3} x^{\nu} [3x^2 + (\nu + 1)(6x^2 - 12\pi x + 4\pi^2)],$$

where $0 < x < \pi$, $\Re \nu > -3/2$. The same result can be obtained by means of Equation (17) for $z = x, y = 1, \Re \nu > -1/2$, taking $\varphi_{\nu} = J_{\nu}, a = 1, b = 0, s = 1, c = 1, F = \zeta, \varepsilon = 0, \delta = 0, d = 0, f = \cos, h = \cos, m = 1, \alpha = \nu + 2$. Similarly, the sums (74.1.20), (74.1.21) and (74.1.22) from [2] are contained in Equation (17).

Example 3 By means of the formula (17) we can find the sums for the series that are not known in the literature. If we consider only finite sums, there are no results for the series with $\alpha - \nu \geq 3, \alpha - \nu \in \mathbb{N}_0$, whereas Equation (17) contains these cases too. For example, for the choice of parameters $\alpha - \nu = 3, \varphi = J, f = \sin, a = 2, b = 1, s = 1$, from the formula (17) one obtains

$$\sum_{n=1}^{\infty} \frac{J_{\nu}((2n - 1)x)}{(2n - 1)^{\nu+3}} \sin(2n - 1)z = \frac{(\pi - z)\pi}{2^{\nu+3}\Gamma(\nu + 1)} x^{\nu} z - \frac{\pi}{2^{\nu+4}\Gamma(\nu + 2)} x^{\nu+2},$$

for $\nu > -1/2$, where the convergence region is K_3 .

Example 4 If we now choose $\alpha - \nu = 4, \varphi = J, f = \cos, a = 1, b = 0, s = -1$, Equation (17) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} J_{\nu}(nx)}{n^{\nu+4}} \cos nz = \frac{(7\pi^4 - 30z^2\pi^2 + 15z^4)x^{\nu}}{45 \cdot 2^{\nu+4}\Gamma(\nu + 1)} + \frac{(3z^2 - \pi^2)x^{\nu+2}}{3 \cdot 2^{\nu+4}\Gamma(\nu + 2)} + \frac{x^{\nu+4}}{2^{\nu+6}\Gamma(\nu + 3)},$$

for $\nu > -1/2$. The convergence region is K_2 .

Example 5 If we finally choose parameters: $\alpha = 5, \nu = 2, \varphi = \mathbf{H}, f = \sin, a = 2, b = 1, s = -1$, we obtain a sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \mathbf{H}_2((2n-1)x)}{(2n-1)^5} \sin(2n-1)z = \frac{1}{30} x^3 z,$$

valid in the convergence region K_4 . Note that this result coming out of the formula (17) cannot be found in the literature for $\alpha - \nu = 3$.

4. Sum of the series (1)

Now, we make use of Equation (16), which is, as we have shown, the formula for finding sum of the series (5), i.e. the right-hand series of Equation (4). Thus we obtain the summation formula of the left-hand side series in Equation (4), which is actually the summation formula for the series (1):

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(s)^{n-1} D_{\nu}((an-b)x)}{(an-b)^{\alpha}} f((an-b)z) \\ &= \frac{(-1)^{\delta(\delta-d)} c \sqrt{\pi} (x/2)^{\nu}}{2\Gamma(\alpha-\nu)h(\pi(\alpha-\nu)/2)} \sum_{j=0}^{\infty} \binom{\alpha-\nu-1}{2j+\delta} z^{\alpha-\nu-2j-1-\delta} x^{2j+\delta} I_{\nu+2j+\delta} \\ &+ \frac{(x/2)^{\nu}}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^{\delta(\delta-d)+i} F(\alpha-\nu-2i-d)}{(2i+d)!} \sum_{j=0}^i \binom{2i+d}{2j+\delta} z^{2i-2j+d-\delta} x^{2j+\delta} I_{\nu+2j+\delta}, \end{aligned} \tag{18}$$

where $D_{\nu} = \left\{ \begin{smallmatrix} B_{\nu,\phi} \\ S_{\nu,\phi} \end{smallmatrix} \right\}$, $g = \left\{ \begin{smallmatrix} \cos \\ \sin \end{smallmatrix} \right\}$, $\delta = \left\{ \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right\}$, $d = \left\{ \begin{smallmatrix} 0 & f=g \\ 1 & f \neq g \end{smallmatrix} \right\}$ and $h = \left\{ \begin{smallmatrix} \cos & f=g \\ \sin & f \neq g \end{smallmatrix} \right\}$, and for the sake of simplicity and brevity, we have denoted $I_{\nu+2j+\delta} = G_j \int_0^1 \phi(y) y^{\nu+2j+\delta} dy$. The parameters a, b, s, c, F as well as convergence regions (which we determined earlier) are read from Table 2.

4.1. Limiting value cases

Very important particular cases of the formula (18) ensue if $h = \sin$ and $\alpha - \nu = 2m$ or $h = \cos$ and $\alpha - \nu = 2m - 1, m \in \mathbb{N}$, when the first term of Equation (18) has zero as a divisor, so we have to deal with a limiting value. We denote $\sigma = \alpha - \nu$ and replace α with $\sigma + \nu$ in Equation (18). Afterwards, choosing, for instance, $a = 2, b = 1, s = 1, c = 1/2, F = \lambda, \varphi_{\nu} = \mathbf{H}_{\nu}, g = \sin, \delta = 1, f = \cos, d = 1, h = \sin$, and finally $\phi(y) = y^{-1}$, we have

$$\begin{aligned} I_{\sigma+\nu}^{S_{\phi}, \cos} &= \frac{\sqrt{\pi} (x/2)^{\nu}}{4\Gamma(\sigma) \sin(\pi\sigma/2)} \sum_{j=0}^{\infty} \binom{\sigma-1}{2j+1} \frac{x^{2j+1} z^{\sigma-2j-2} j!}{(\nu+2j+1)\Gamma(j+\nu+3/2)} \\ &+ \frac{(x/2)^{\nu}}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i \lambda (\sigma-2i-1)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j+1} \frac{x^{2j+1} z^{2i-2j} j!}{(\nu+2j+1)\Gamma(j+\nu+3/2)}, \end{aligned}$$

so that we find

$$\begin{aligned} \Phi_{2m,v}(x,z) &= \lim_{\sigma \rightarrow 2m} \left[\frac{\sqrt{\pi}(x/2)^v}{4\Gamma(\sigma)\sin(\pi\sigma/2)} \sum_{j=0}^{m-1} \binom{\sigma-1}{2j+1} \frac{x^{2j+1}z^{\sigma-2j-2}j!}{(v+2j+1)\Gamma(j+v+3/2)} \right. \\ &\quad \left. + \frac{(x/2)^v}{\sqrt{\pi}} \sum_{i=0}^{m-1} \frac{(-1)^i\lambda(\sigma-2i-1)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j+1} \frac{x^{2j+1}z^{2i-2j}j!}{(v+2j+1)\Gamma(j+v+3/2)} \right] \\ &= \frac{(x/2)^v}{\sqrt{\pi}} \left[\frac{(-1)^m}{2} \sum_{i=0}^{m-1} \frac{(\log(z/2) - \psi(2m-2i-1) - \gamma)x^{2i+1}z^{2m-2i-2}i!}{(v+2i+1)(2m-2i-2)!(2i+1)!\Gamma(v+i+3/2)} \right. \\ &\quad \left. + \sum_{i=0}^{m-2} \frac{(-1)^i\lambda(2m-2i-1)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j+1} \frac{x^{2j+1}z^{2i-2j}j!}{(v+2j+1)\Gamma(v+j+3/2)} \right], \end{aligned}$$

and

$$\begin{aligned} R_{2m,v}(x,z) &= \lim_{\sigma \rightarrow 2m} \frac{\sqrt{\pi}(x/2)^v}{4\Gamma(\sigma)\sin(\pi\sigma/2)} \sum_{j=m}^{\infty} \binom{\sigma-1}{2j+1} \frac{x^{2j+1}z^{\sigma-2j-2}j!}{(v+2j+1)\Gamma(j+v+3/2)} \\ &= \frac{(-1)^m(x/2)^v}{2(2m)!\sqrt{\pi}} \sum_{j=m}^{\infty} \frac{x^{2j+1}z^{2m-2j-2}j!}{\binom{2j+1}{2m}(v+2j+1)\Gamma(v+j+3/2)}. \end{aligned}$$

Finally, the series involving the product of a Struve integral and cosine is as follows:

$$\begin{aligned} I_{2m+v}^{S_{\phi},\cos} &= \sum_{n=1}^{\infty} \frac{\cos(2n-1)z}{(2n-1)^{2m+v}} \int_0^1 \frac{\mathbf{H}_v((2n-1)xy)}{y} dy = \Phi_{2m,v}(x,z) + R_{2m,v}(x,z) \\ &\quad + \frac{(x/2)^v}{\sqrt{\pi}} \sum_{i=m}^{\infty} \frac{(-1)^i\lambda(2m-2i-1)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j+1} \frac{x^{2j+1}z^{2i-2j}j!}{(v+2j+1)\Gamma(v+j+3/2)} \end{aligned}$$

where $(x,z) \in K_3$ (see Table 2 on the page 4).

Example 6 If we choose $m=2$, then we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\cos(2n-1)z}{(2n-1)^{4+v}} \int_0^1 \frac{\mathbf{H}_v((2n-1)xy)}{y} dy \\ &= \frac{(x/2)^v}{\sqrt{\pi}} \left[\frac{x((\log(z/2) - (3/2))z^2 + (7/2)\zeta(3))}{4(v+1)\Gamma(v+3/2)} \right. \\ &\quad + \frac{1}{48} \sum_{j=2}^{\infty} \frac{x^{2j+1}z^{2-2j}j!}{\binom{2j+1}{4}(v+2j+1)\Gamma(v+j+3/2)} + \frac{x^3 \log(z/2)}{12(v+3)\Gamma(v+5/2)} \\ &\quad \left. + \sum_{i=2}^{\infty} \frac{(-1)^i\lambda(3-2i)}{(2i+1)!} \sum_{k=0}^i \binom{2i+1}{2k+1} \frac{x^{2k+1}z^{2i-2k}k!}{(v+2k+1)\Gamma(k+v+3/2)} \right]. \end{aligned}$$

Similarly, we can obtain the other particular cases, without having previously to calculate the integral involved.

4.2. Closed form cases

The infinite series (18) is brought in closed form if $F = \zeta, \eta, \lambda$ and $\alpha - \nu - d = 2m$ or $F = \beta$ and $\alpha - \nu - d = 2m - 1$ ($m \in \mathbb{N}$), so that we write $\alpha = \nu + 2m + d - \varepsilon$, and have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(s)^{n-1} D_{\nu}((an - b)x)}{(an - b)^{\nu+2m+d-\varepsilon}} f((an - b)z) \\ &= \frac{(-1)^{\delta(\delta-d)} c \sqrt{\pi} (x/2)^{\nu}}{2\Gamma(2m + d - \varepsilon) h(m\pi + \pi(d - \varepsilon)/2)} \\ & \quad \times \sum_{j=0}^m \binom{2m + d - \varepsilon - 1}{2j + \delta} z^{2m+d-\varepsilon-2j-1-\delta} x^{2j+\delta} I_{\nu+2j+\delta} \\ & \quad + \frac{(x/2)^{\nu}}{\sqrt{\pi}} \sum_{i=0}^m \frac{(-1)^{\delta(\delta-d)+i} F(2m - 2i - \varepsilon)}{(2i + d)!} \\ & \quad \times \sum_{j=0}^i \binom{2i + d}{2j + \delta} z^{2i-2j+d-\delta} x^{2j+\delta} I_{\nu+2j+\delta}, \end{aligned} \tag{19}$$

where $D_{\nu} = \begin{Bmatrix} B_{\nu, \phi} \\ S_{\nu, \phi} \end{Bmatrix}$, $g = \begin{Bmatrix} \cos \\ \sin \end{Bmatrix}$, $\delta = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$, $d = \begin{Bmatrix} 0 & f=g \\ 1 & f \neq g \end{Bmatrix}$ and $h = \begin{Bmatrix} \cos & f=g \\ \sin & f \neq g \end{Bmatrix}$, $\varepsilon = 0$ if $F = \zeta, \eta, \lambda$ and $\varepsilon = 1$ if $F = \beta$. The parameters a, b, s, c, F and convergence regions are read from Table 2.

Example 7 First we take $a = 1, b = 0, s = 1$ in Equation (19), there follows $c = 1$ and $F = \zeta$ (see Table 2), then $\varepsilon = 0$. For $D_2 = S_{2, \phi}$, there must be $g = \sin$, which means $\delta = 1$. If we take $f = \cos$, then $h = \sin$ and $d = 1$ because $f \neq g$. We choose $\phi(y) = \text{ctg } y$. Let $m = 1$. The function $\text{ctg } y$ is unbounded in the neighbourhood of 0 and the integral $\int_0^1 \text{ctg } y \, dy$ does not converge, so we cannot apply Lemma 1. However, for $0 < |x| < \pi$, $\lim_{y \rightarrow 0+} \text{ctg } y \mathbf{H}_2(nxy) = 0$, where $n \in \mathbb{N}$, the function $\text{ctg } y \mathbf{H}_2(nxy)$ is integrable with respect to $y \in (0, 1)$, and we have

$$\begin{aligned} \left| x \int_0^1 \text{ctg } y \mathbf{H}_2(nxy) \, dy \right| &= \left| \int_0^x \text{ctg } \frac{t}{x} \mathbf{H}_2(nt) \, dt \right| \leq \int_0^x \left| \text{ctg } \frac{t}{x} \mathbf{H}_2(nt) \right| \, dt \\ &\leq \int_0^{\pi} \left| \text{ctg } \frac{t}{\pi} \mathbf{H}_2(nt) \right| \, dt. \end{aligned}$$

Also, $|\text{tg}(t/\pi)| > |t|/\pi$ implies $|\text{ctg}(t/\pi)| < \pi/|t|$, for $|t| < \pi$. Additionally, we find

$$|\mathbf{H}_2(nt)| = \frac{2n|t|}{3\pi} \left| 1 - \frac{3\pi J_1(nt)}{2nt} + \frac{3\pi J_2(nt)}{n^2 t^2} \right| < \frac{2n|t|}{3\pi} \left(1 + \frac{3\pi}{8} \right)$$

So,

$$\left| \text{ctg } \frac{t}{\pi} \mathbf{H}_2(nt) \right| = \left| \text{ctg } \frac{t}{\pi} \right| \cdot |\mathbf{H}_2(nt)| < \frac{\pi}{|t|} \cdot \frac{2n|t|}{3\pi} \left(1 + \frac{3\pi}{8} \right) = \frac{2n}{3} \left(1 + \frac{3\pi}{8} \right),$$

and there follows

$$\int_0^{\pi} \left| \text{ctg } \frac{t}{\pi} \mathbf{H}_2(nt) \right| \, dt < \frac{2n}{3} \left(1 + \frac{3\pi}{8} \right) \int_0^{\pi} \, dt = \frac{2n\pi}{3} \left(1 + \frac{3\pi}{8} \right),$$

which means that $\left| \int_0^1 \text{ctg } y \mathbf{H}_2(nxy) \, dy \right| < n\pi(8 + 3\pi)/12|x| = M_n(x)$, and we can easily see that for each $x, 0 < |x| < \pi$, the sequence $M_n(x)/n^3 \rightarrow 0$ monotonically, so that Lemma 3 may be

applied. Making use of Equation (13), we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\cos nz}{n^5} \int_0^1 \operatorname{ctg} y \mathbf{H}_2(nxy) dy \\ &= \frac{x^3}{1156\sqrt{\pi}} ((8\pi^2 - 24\pi z + 12z^2) \log(2 \sin 1) + (12\pi^2 - 36\pi z + 18z^2)(\operatorname{Cl}_2(2) + \operatorname{Cl}_3(2)) \\ & \quad - (6\pi^2 - 18\pi z + 9z^2)\operatorname{Cl}_4(2)) + \frac{x^5}{4608\sqrt{\pi}} (\log(2 \sin 1) + 10\operatorname{Cl}_2(2) + 20\operatorname{Cl}_3(2) \\ & \quad - 30(\operatorname{Cl}_4(2) + \operatorname{Cl}_5(2)) + 15\operatorname{Cl}_6(2)), \end{aligned}$$

where $|x| < \pi$ and $|x| < z < 2\pi - |x|$ (see Table 2). On the right-hand side are Clausen functions defined by [1]

$$\operatorname{Cl}_{2\nu}(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2\nu}}, \quad \operatorname{Cl}_{2\nu-1}(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2\nu-1}}, \quad \nu \in \mathbb{N}.$$

Example 8 Let $a = 1, b = 0, s = -1$ in Equation (19), implying $c = 0$ and $F = \eta, \varepsilon = 0$. For $D_{1/2} = B_{1/2, \phi}$ there must be $g = \cos$ and $\delta = 0$. Further, we take $f = \cos$ implying $h = \cos, d = 0$ because $f = g$. Let $m = 2$. We choose $\phi(y) = (1 - y^2)^{-1/2}$. It is unbounded about 1, but integrable on $(0, 1)$, thus satisfying conditions of Lemma 2. So applying Equation (19), we find

$$\begin{aligned} & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nz}{n^{9/2}} \int_0^1 \frac{J_{1/2}(nxy)}{\sqrt{1 - y^2}} dy \\ &= \frac{\pi \sqrt{x\pi}}{4\Gamma^2(1/4)} \left(\frac{7\pi^4}{90} - \frac{\pi^2 z^2}{3} + \frac{z^4}{6} - \frac{\pi^2 x^2}{20} + \frac{3x^2 z^2}{20} + \frac{7x^4}{720} \right), \end{aligned}$$

where $|x| < \pi$ and $|x| - \pi < z < \pi - |x|$ (see Table 2).

We have already said that in order to find the sum of the series (1), it is not necessary to calculate the integrals (2). Besides, it does not have to be done elementarily. Yet, if we calculate the integral in Example 8, the above series takes a different form, giving rise to the following formula

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nz}{n^{9/2}} J_{1/4}^2\left(\frac{nx}{2}\right) = \frac{\sqrt{x\pi}}{2\Gamma^2(1/4)} \left(\frac{7\pi^4}{90} - \frac{\pi^2 z^2}{3} + \frac{z^4}{6} - \frac{\pi^2 x^2}{20} + \frac{3x^2 z^2}{20} + \frac{7x^4}{720} \right),$$

whereby we obtain the sum of a new series. Generally speaking, for $\nu > -1$, there holds

$$\int_0^1 \frac{J_{\nu}(nxy)}{\sqrt{1 - y^2}} dy = \frac{\pi}{2} J_{\nu/2}^2\left(\frac{nx}{2}\right),$$

so for this type of integrals there exists a whole class of new closed form formulas.

Example 9 Further, we take $a = 2, b = 1, s = 1$ in Equation (19). In Table 2, we read $c = 1/2$ and $F = \lambda$. So $\varepsilon = 0$. If we choose $D_{1/3} = S_{1/3, \phi}, f = \sin$, then we have $g = \cos, \delta = 0, d = 1, h = \sin$. Let $m = 1$ and $\phi(y) = \log y$, which is unbounded in the neighbourhood of 0, but integrable on $(0, 1)$, so Lemma 2 holds. Applying Equation (19), we obtain

$$I_1^{S_{\phi}, \sin} = \sum_{n=1}^{\infty} \frac{\sin(2n - 1)z}{(2n - 1)^{10/3}} \int_0^1 \log y \mathbf{H}_{1/3}((2n - 1)xy) dy = \frac{27\sqrt[3]{x}\Gamma(1/6)}{640\sqrt[3]{2}} \left(z\pi + z^2 + \frac{12x^2}{275} \right),$$

where $|x| < \pi/2$ and $|x| < z < \pi - |x|$ (see Table 2).

Example 10 Finally, let in Equation (19) be $a = 2$, $b = 1$, $s = -1$, implying $c = 0$, $F = \beta$, $\varepsilon = 1$. If we choose $D_1 = B_{1, \phi}$, there follows $g = \cos$ and $\delta = 0$. If $f = \sin$, then $d = 1$ and $h = \sin$ because $f \neq g$. Further, let $\phi(y) = y^\mu$, $\mu > -2$, and $m = 1$. The integral $\int_0^1 y^\mu dy$ does not necessarily converge for $\mu > -2$. However, making use of Equation (6), where we substitute u for $\cos t$, we come, for $|x| < \pi/2$, to the following estimate

$$y^\mu |J_1((2n-1)xy)| \leq \frac{2|x|}{\pi} (2n-1)y^{\mu+1} \left| \frac{\sin((2n-1)xy)}{(2n-1)xy} \right| < (2n-1)y^{\mu+1},$$

whereupon we find

$$\left| \int_0^1 J_1((2n-1)xy)y^\mu dy \right| \leq \int_0^1 |J_1((2n-1)xy)|y^\mu dy < (2n-1) \int_0^1 y^{\mu+1} dy = \frac{2n-1}{\mu+2},$$

with $M_n(x) = (2n-1)/(\mu+2)$, so $M_n(x)/(2n-1)^3$ monotonically tends to zero when n increases to infinity, which means that Lemma 3 holds, and by applying Equation (19), we obtain

$$I_3^{B_\phi, \sin} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(2n-1)z}{(2n-1)^3} \int_0^1 J_1((2n-1)xy) y^\mu dy = \frac{\pi x z}{8(\mu+2)},$$

$|x| < \pi/2$ and $|x| - \pi/2 < z < \pi/2 - |x|$ (see Table 2).

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References

- [1] M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables*, Dover Publications, New York, 1972.
- [2] E.R. Hansen, *A Table of Series and Products*, Prentice-Hall Inc, Englewood Cliffs, N.J., 1975.
- [3] K. Knopp, *Theory and Application of Infinite Series*, Dover Publications, New York, 1990.
- [4] A.P. Prudnikov, Y.A. Brychkov, and O.I. Marichev, *Integrals and Series, Vol. 1: Elementary Functions*, Gordon and Breach Science Publishers, New York, 1986.
- [5] S.B. Tričković, M.V. Vidanović, and M.S. Stanković, *Series involving the product of a trigonometric integral and a trigonometric function*, Integral Transforms Spec. Func. 18(10) (2007), pp. 751–763.
- [6] S.B. Tričković, M.V. Vidanović, and M.S. Stanković, *On the summation of trigonometric series*, Integral Transforms Spec. Func. 19(6) (2008), pp. 441–452.