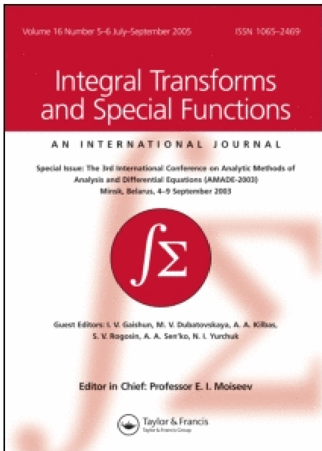

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On the summation of trigonometric series

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We deal with the series

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^{\alpha}}, \quad \alpha \in \mathbb{R}^{+},$$

and express it as a power series in terms of Riemann's ζ or Catalan's β function or Dirichlet functions η and λ. Also, closed form cases as well as those when it is necessary to take limit have been thoroughly analyzed. Some applications such as convergence acceleration are considered too.

Keywords: Riemann's ζ and Catalan's β function; Dirichlet η and λ functions

2000 Mathematics Subject Classification: Primary: 33C10; Secondary: 11M06, 65B10

1. Introduction and preliminaries

There have been many particular cases (see, for example, [2] and [7]) of the trigonometric series of the type

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1}\ f((an-b)x)}{(an-b)^{\alpha}}, \quad \alpha \in \mathbb{R}^{+}, \tag{1}$$

where s is 1 or -1, $f = \sin$ or $f = \cos$ and $a = \{{}^{1}_{2}\} b = \{{}^{0}_{1}\}$. We first derive a general formula for finding the sum of Equation (1), meaning that we consider α to be a positive real parameter with the exclusion of positive integers. Afterwards we regard α as a positive integer, whereupon there appears a necessity to distinguish between the cases when infinite series (1) reduces to a finite sum, and those when limiting value must be taken. Apart from all known particular cases, the general formula yields new ones. We start with the following

LEMMA 1 *Series* (1) *is uniformly convergent for* $\alpha > 0$. *Convergence regions are given in Table* 1, *where* ζ *is Riemann's zeta function* $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$, η *and* λ *are Dirichlet functions* $\eta(z) = \sum_{k=1}^{\infty}(-1)^{k-1}k^{-z} = (1-2^{1-z})\zeta(z)$, $\lambda(z) = \sum_{k=0}^{\infty}(2k+1)^{-z} = (1-2^{-z})\zeta(z)$, *and* β *is Catalan's beta function* $\beta(z) = \sum_{k=0}^{\infty}(-1)^{k}(2k+1)^{-z}$.

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Table 1. Parameters and convergence region

a	b	s	c	F	f	δ	r	$\alpha \in$	Convergence region
1	0	1	1	ζ	sin	1	$2m-1$	$\mathbb{R}^+ \setminus 2\mathbb{N}$	$0 < x < 2\pi$
					cos	0	$2m$	$\mathbb{R}^+ \setminus 2\mathbb{N}-1$	
		-1	0	η	sin	1	$2m-1$	$\mathbb{R}^+ \setminus 2\mathbb{N}$	$-\pi < x < \pi$
					cos	0	$2m$	$\mathbb{R}^+ \setminus 2\mathbb{N}-1$	
2	1	1	1/2	λ	sin	1	$2m-1$	$\mathbb{R}^+ \setminus 2\mathbb{N}$	$0 < x < \pi$
					cos	0	$2m$	$\mathbb{R}^+ \setminus 2\mathbb{N}-1$	
		-1	0	β	sin	1	$2m$	$\mathbb{R}^+ \setminus 2\mathbb{N}-1$	$-\pi/2 < x < \pi/2$
					cos	0	$2m-1$	$\mathbb{R}^+ \setminus 2\mathbb{N}$	

Remark 1 We note that ζ, η, λ are analytic in the whole complex plane except for $z = 1$, where they have a pole, whereas Catalan's beta function $\beta(z) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-z}$ satisfies the functional equation $\beta(z) = (\pi/2)^{z-1}\Gamma(1-z)\cos(\pi z/2)\beta(1-z)$ extending the beta function to the left side of the complex plane $Re\, z < 1$.

Proof We shall make use of Dirichlet's test saying that (see [3]) the series $\sum_{n=0}^{\infty} a_n(x)b_n(x)$ is uniformly convergent in D, if the partial sums of $\sum_{n=0}^{\infty} a_n(x)$ are uniformly bounded in D, and the sequence $b_n(x)$, being monotonic for every fixed x, uniformly converges to 0.

Let us suppose first that $a = 1, b = 0$. If $s = 1$, then there holds

$$\left|\sum_{k=1}^{n} f(kx)\right| \le \frac{1}{\sin(x/2)} \le \frac{1}{\sin\varepsilon}$$

for $0 < x < 2\pi$, because $\sin(x/2) \ge \sin\varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \le x/2 \le \pi - \varepsilon$, meaning that the partial sums are uniformly bounded with respect to $x \in (0, 2\pi)$. Similarly, if $s = -1$, we would have

$$\left|\sum_{k=1}^{n}(-1)^{k-1} f(kx)\right| \le \frac{1}{\cos(x/2)} \le \frac{1}{\sin\varepsilon}$$

for $-\pi < x < \pi$, because $\cos(x/2) \ge \sin\varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-(\pi/2) + \varepsilon \le (x/2) \le (\pi/2) - \varepsilon$. Here the partial sums are uniformly bounded with respect to $x \in (-\pi, \pi)$.

We now suppose $a = 2, b = 1$. If $s = 1$, then

$$\left|\sum_{k=1}^{n} f((2k-1)x)\right| \le \frac{1}{\sin x} \le \frac{1}{\sin\varepsilon}$$

for $0 < x < \pi$, because $\sin x \ge \sin\varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \le x \le \pi - \varepsilon$. The partial sums are uniformly bounded with respect to $x \in (0, \pi)$. Finally if $s = -1$, then

$$\left|\sum_{k=1}^{n}(-1)^{k-1} f((2k-1)x)\right| \le \frac{1}{\cos x} \le \frac{1}{\sin\varepsilon}$$

for $-(\pi/2) < x < (\pi/2)$, because $\cos x \ge \sin\varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-(\pi/2) + \varepsilon \le x \le (\pi/2) - \varepsilon$, and the partial sums are uniformly bounded with respect to $x \in (-(\pi/2), (\pi/2))$. Since in each of the particular cases the sequence $1/(an - b)^{\alpha}$ tends to 0 for $\alpha > 0$, by virtue of Dirichlet's test all the series in question uniformly converge, whereby the proof is complete. ∎

2. The general formula

THEOREM 1 *For the values of* α *and the other relevant parameters from Table* 1, *there holds*

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^{\alpha}} = \frac{c\pi x^{\alpha-1}}{2\Gamma(\alpha) f(\pi\alpha/2)} + \sum_{k=0}^{\infty} \frac{(-1)^k F(\alpha-2k-\delta)}{(2k+\delta)!} x^{2k+\delta}. \quad (2)$$

Proof We make use of the *polylogarithm* $\mathrm{Li}_\alpha(z)$ defined by the series (see [4]), and with the following integral representation

$$\mathrm{Li}_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1}}{\mathrm{e}^t/z - 1} \mathrm{d}t,$$

where the right-hand side integral converges for $z \in \mathbb{C} \setminus \{z \mid z \in \mathbb{R}, z \geq 1\}$, and it is referred to as *Bose's integral*. So for $\alpha > 0$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin nx}{n^\alpha} &= \frac{i}{2} \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-inx} - \mathrm{e}^{inx}}{n^\alpha} = \frac{i}{2} \left(\mathrm{Li}_\alpha(\mathrm{e}^{-ix}) - \mathrm{Li}_\alpha(\mathrm{e}^{ix})\right), \\ \sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha} &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-inx} + \mathrm{e}^{inx}}{n^\alpha} = \frac{1}{2} \left(\mathrm{Li}_\alpha(\mathrm{e}^{-ix}) + \mathrm{Li}_\alpha(\mathrm{e}^{ix})\right) \end{aligned} \quad (3)$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n^\alpha} &= \frac{i}{2} \left(\mathrm{Li}_\alpha(-\mathrm{e}^{ix}) - \mathrm{Li}_\alpha(-\mathrm{e}^{-ix})\right), \\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^\alpha} &= -\frac{1}{2} \left(\mathrm{Li}_\alpha(-\mathrm{e}^{-ix}) + \mathrm{Li}_\alpha(-\mathrm{e}^{ix})\right). \end{aligned} \quad (4)$$

Also, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^\alpha} &= \frac{i}{2} \left(\left(\mathrm{Li}_\alpha(\mathrm{e}^{-ix}) - \frac{1}{2^\alpha} \mathrm{Li}_\alpha(\mathrm{e}^{-2ix}) \right) - \left(\mathrm{Li}_\alpha(\mathrm{e}^{ix}) - \frac{1}{2^\alpha} \mathrm{Li}_\alpha(\mathrm{e}^{2ix}) \right) \right), \\ \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^\alpha} &= \frac{1}{2} \left(\left(\mathrm{Li}_\alpha(\mathrm{e}^{-ix}) - \frac{1}{2^\alpha} \mathrm{Li}_\alpha(\mathrm{e}^{-2ix}) \right) + \left(\mathrm{Li}_\alpha(\mathrm{e}^{ix}) - \frac{1}{2^\alpha} \mathrm{Li}_\alpha(\mathrm{e}^{2ix}) \right) \right) \end{aligned} \quad (5)$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(2n-1)x}{(2n-1)^\alpha} &= \frac{1}{4} \left(\mathrm{Li}_\alpha(-i\mathrm{e}^{ix}) - \mathrm{Li}_\alpha(-i\mathrm{e}^{-ix}) - \mathrm{Li}_\alpha(i\mathrm{e}^{ix}) + \mathrm{Li}_\alpha(i\mathrm{e}^{-ix})\right), \\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(2n-1)x}{(2n-1)^\alpha} &= \frac{i}{4} \left(\mathrm{Li}_\alpha(-i\mathrm{e}^{ix}) + \mathrm{Li}_\alpha(-i\mathrm{e}^{-ix}) - \mathrm{Li}_\alpha(i\mathrm{e}^{ix}) - \mathrm{Li}_\alpha(i\mathrm{e}^{-ix})\right). \end{aligned} \quad (6)$$

Here, we note that in order to obtain the right-hand sides of Equation (4) and (6), we have taken advantage of the representation $(-1)^n = \cos n\pi$, and then, by means of elementary trigonometric identities, we split up the product into the sum of two trigonometric functions.

We shall now consider the Mellin transform of the polylogarithm in the form of Bose's integral. The Mellin transform of a function f and the inverse transform of a function φ are (see [6])

$$M(f(x)) = \int_0^\infty x^{u-1} f(x)\mathrm{d}x, \qquad M^{-1}(\varphi(u)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-u}\varphi(u)\mathrm{d}u.$$

This integral transform is closely connected to the theory of Dirichlet series and is often used in number theory and the theory of asymptotic expansions. Also, it is closely related to the Laplace and Fourier transform as well as to the theory of the gamma function and allied special functions. So we find

$$M(\mathrm{Li}_\alpha(p\,\mathrm{e}^{-x})) = \int_0^\infty x^{u-1}\mathrm{Li}_\alpha(p\,\mathrm{e}^{-x})\mathrm{d}x = \frac{1}{\Gamma(\alpha)}\int_0^\infty\int_0^\infty \frac{t^{\alpha-1}x^{u-1}}{\mathrm{e}^{t+x}/p-1}\mathrm{d}t\mathrm{d}x.$$

The change of variables $x = ab$, $t = a(1-b)$ allows the integrals to be separated

$$M(\mathrm{Li}_\alpha(p\,\mathrm{e}^{-x})) = \frac{1}{\Gamma(\alpha)}\int_0^1 b^{u-1}(1-b)^{\alpha-1}\mathrm{d}b\int_0^\infty \frac{a^{\alpha+u-1}}{\mathrm{e}^a/p-1}\mathrm{d}a = \Gamma(u)\mathrm{Li}_{\alpha+u}(p). \tag{7}$$

For $p = 1$, because $\mathrm{Li}_{\alpha+u}(1) = \zeta(\alpha+u)$, through the inverse Mellin transform, we have

$$\mathrm{Li}_\alpha(\mathrm{e}^{-x}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(u)\zeta(\alpha+u)x^{-u}\mathrm{d}u,$$

where c is a constant to the right of the poles of the integrand. The path of integration may be converted into a closed contour, and the poles of the integrand are those of $\Gamma(u)$ at $u = 0, -1, -2, \dots$, and of $\zeta(\alpha+u)$ at $u = 1-\alpha$. Summing the residues yields a representation of the polylogarithm as a power series

$$\mathrm{Li}_\alpha(\mathrm{e}^\mu) = (-\mu)^{\alpha-1}\Gamma(1-\alpha) + \sum_{k=0}^\infty \frac{\zeta(\alpha-k)}{k!}\mu^k, \quad |\mu| < 2\pi, \quad \alpha \neq 1, 2, 3, \dots \tag{8}$$

about $\mu = 0$. Further, following Equation (3), we have

$$\begin{aligned}\frac{i}{2}(\mathrm{Li}_\alpha(\mathrm{e}^{-\mu}) - \mathrm{Li}_\alpha(\mathrm{e}^\mu)) &= \frac{i}{2}(\mu^{\alpha-1} - (-\mu)^{\alpha-1})\Gamma(1-\alpha) - i\sum_{k=0}^\infty \frac{\zeta(\alpha-2k-1)}{(2k+1)!}\mu^{2k+1},\\ \frac{1}{2}(\mathrm{Li}_\alpha(\mathrm{e}^\mu) + \mathrm{Li}_\alpha(\mathrm{e}^{-\mu})) &= \frac{1}{2}((-\mu)^{\alpha-1} + \mu^{\alpha-1})\Gamma(1-\alpha) + \sum_{k=0}^\infty \frac{\zeta(\alpha-2k)}{(2k)!}\mu^{2k},\end{aligned} \tag{9}$$

where we replace μ with ix, $0 < x < 2\pi$. For a positive real non-integer α, the gamma function is finite, and so in view of

$$(\pm i)^{\alpha-1} = \mathrm{e}^{\pm i(\alpha-1)\pi/2} = \cos\frac{\pi}{2}(\alpha-1) \pm i\sin\frac{\pi}{2}(\alpha-1), \tag{10}$$

we calculate

$$\begin{aligned}\frac{i}{2}(\mu^{\alpha-1} - (-\mu)^{\alpha-1})\Gamma(1-\alpha) &= \frac{i\pi x^{\alpha-1}}{2\Gamma(\alpha)\sin\pi\alpha}(i^{\alpha-1} - (-i)^{\alpha-1}) = \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha)\sin(\pi/2)\alpha},\\ \frac{1}{2}((-\mu)^{\alpha-1} + \mu^{\alpha-1})\Gamma(1-\alpha) &= \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha)\sin\pi\alpha}((-i)^{\alpha-1} + i^{\alpha-1}) = \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha)\cos(\pi/2)\alpha},\end{aligned}$$

which, in conjunction with Equation (9) gives the sums of the series in Equation (3), *i.e.*

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^{\alpha}} = \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha)\sin(\pi/2)\alpha} + \sum_{k=0}^{\infty}\frac{(-1)^k\zeta(\alpha-2k-1)}{(2k+1)!}\,x^{2k+1},$$
$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^{\alpha}} = \frac{\pi x^{\alpha-1}}{2\Gamma(\alpha)\cos(\pi/2)\alpha} + \sum_{k=0}^{\infty}\frac{(-1)^k\zeta(\alpha-2k)}{(2k)!}\,x^{2k}.$$

We now consider Equation (4). Because of Equation (7), we find $M(\mathrm{Li}_\alpha(-\mathrm{e}^{-x})) = -\Gamma(u)\eta(\alpha+u)$, since $\mathrm{Li}_{\alpha+u}(-1) = -\eta(\alpha+u)$. Through the inverse Mellin transform and conversion of path of integration into a closed contour, after summing the residues at $u = 0, -1, -2, \ldots$ and $u = 1-\alpha$ (which is 0), we have

$$\mathrm{Li}_\alpha(-\mathrm{e}^{\mu}) = -\sum_{k=0}^{\infty}\frac{\eta(\alpha-k)}{k!}\,\mu^k, \quad |\mu| < 2\pi, \quad \alpha \neq 1, 2, 3, \ldots,$$

whence we find

$$\frac{i}{2}(\mathrm{Li}_\alpha(-\mathrm{e}^{-\mu}) - \mathrm{Li}_\alpha(-\mathrm{e}^{\mu})) = i\sum_{k=0}^{\infty}\frac{\eta(\alpha-2k-1)}{(2k+1)!}\,\mu^{2k+1},$$
$$-\frac{1}{2}(\mathrm{Li}_\alpha(-\mathrm{e}^{\mu}) + \mathrm{Li}_\alpha(-\mathrm{e}^{-\mu})) = \sum_{k=0}^{\infty}\frac{\eta(\alpha-2k)}{(2k)!}\,\mu^{2k}, \tag{11}$$

and for $\mu = ix$ $(-\pi < x < \pi)$, we obtain the sums of the series in Equation (4), *i.e.*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin nx}{n^{\alpha}} = \sum_{k=0}^{\infty}\frac{(-1)^k\eta(\alpha-2k-1)}{(2k+1)!}\,x^{2k+1},$$
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos nx}{n^{\alpha}} = \sum_{k=0}^{\infty}\frac{(-1)^k\eta(\alpha-2k)}{(2k)!}\,x^{2k}.$$

As regards Equation (5), judging by its structure, we have to repeat the procedure as that for obtaining Equation (8), *i.e.* once for $\mathrm{Li}_\alpha(\mathrm{e}^{\pm\mu})$ $(|\mu| < 2\pi)$, and once again for $\mathrm{Li}_\alpha(\mathrm{e}^{\pm 2\mu})$ $(|\mu| < \pi)$, so that we have

$$\frac{i}{2}\left(\left(\mathrm{Li}_\alpha(\mathrm{e}^{-\mu}) - \frac{1}{2^\alpha}\mathrm{Li}_\alpha(\mathrm{e}^{-2\mu})\right) - \left(\mathrm{Li}_\alpha(\mathrm{e}^{\mu}) - \frac{1}{2^\alpha}\mathrm{Li}_\alpha(\mathrm{e}^{2\mu})\right)\right)$$
$$= \frac{i}{4}(\mu^{\alpha-1} - (-\mu)^{\alpha-1})\Gamma(1-\alpha) - i\sum_{k=0}^{\infty}\frac{\lambda(\alpha-2k-1)}{(2k+1)!}\,\mu^{2k+1},$$
$$\frac{1}{2}\left(\left(\mathrm{Li}_\alpha(\mathrm{e}^{-\mu}) - \frac{1}{2^\alpha}\mathrm{Li}_\alpha(\mathrm{e}^{-2\mu})\right) + \left(\mathrm{Li}_\alpha(\mathrm{e}^{\mu}) - \frac{1}{2^\alpha}\mathrm{Li}_\alpha(\mathrm{e}^{2\mu})\right)\right)$$
$$= \frac{1}{4}(\mu^{\alpha-1} + (-\mu)^{\alpha-1})\Gamma(1-\alpha) + \sum_{k=0}^{\infty}\frac{\lambda(\alpha-2k)}{(2k)!}\,\mu^{2k}, \quad |\mu| < \pi, \quad \alpha \neq 1, 2, 3, \ldots, \tag{12}$$

where we have taken into account $\lambda(z) = (1 - 2^{-z})\zeta(z)$. Thus, for $\mu = ix$ $(0 < x < \pi)$, we obtain the sums of the series in Equation (5), *i.e.*

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^{\alpha}} = \frac{\pi x^{\alpha-1}}{4\Gamma(\alpha)\sin(\pi/2)\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k \lambda(\alpha-2k-1)}{(2k+1)!} x^{2k+1},$$
$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{\alpha}} = \frac{\pi x^{\alpha-1}}{4\Gamma(\alpha)\cos(\pi/2)\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k \lambda(\alpha-2k)}{(2k)!} x^{2k}.$$

Finally, in the case of Equation (6), we first calculate $\mathrm{Li}_{\alpha+u}(\pm i) = 2^{-(\alpha+u)}\eta(\alpha+u) \pm i\beta(\alpha+u)$ and find $M(\mathrm{Li}_{\alpha}(\pm i \mathrm{e}^{-x})) = \Gamma(u)\left(2^{-(\alpha+u)}\eta(\alpha+u) \pm i\beta(\alpha+u)\right)$. Applying again the inverse Mellin transform and converting the path of integration into closed contour, we evaluate the residues at $u = 0, -1, -2, \dots$ and $u = 1 - \alpha$ (which is 0), so that after their summation, a rearrangement and simplification, we obtain

$$\begin{aligned}\frac{1}{4}\left(\mathrm{Li}_{\alpha}(-i\mathrm{e}^{\mu}) - \mathrm{Li}_{\alpha}(-i\mathrm{e}^{-\mu}) - \mathrm{Li}_{\alpha}(i\mathrm{e}^{\mu}) + \mathrm{Li}_{\alpha}(i\mathrm{e}^{-\mu})\right) &= -i\sum_{k=0}^{\infty} \frac{\beta(\alpha-2k-1)}{(2k+1)!}\mu^{2k+1},\\ \frac{i}{4}\left(\mathrm{Li}_{\alpha}(-i\mathrm{e}^{\mu}) + \mathrm{Li}_{\alpha}(-i\mathrm{e}^{-\mu}) - \mathrm{Li}_{\alpha}(i\mathrm{e}^{\mu}) - \mathrm{Li}_{\alpha}(i\mathrm{e}^{-\mu})\right) &= \sum_{k=0}^{\infty} \frac{\beta(\alpha-2k)}{(2k)!}\mu^{2k},\end{aligned} \tag{13}$$

where $|\mu| < \pi/2, \alpha \neq 1, 2, 3, \dots$. For $\mu = ix(-\pi/2 < x < \pi/2)$, we find the sums of the series in Equation (6), *i.e.*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(-1)^k \beta(\alpha-2k-1)}{(2k+1)!} x^{2k+1},$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(-1)^k \beta(\alpha-2k)}{(2k)!} x^{2k}.$$

Gathering all these results, we conclude that Equation (2) holds. ■

3. Closed form cases

We shall express now, for certain values of parameters, series (1) as polynomials, saying then that we have brought the infinite series in *closed form*.

THEOREM 2 *For the values of r and the other relevant parameters from Table* 1, *there holds*

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f(an-b)x)}{(an-b)^r} = \frac{(-1)^{[r/2]} c\pi x^{r-1}}{2(r-1)!} + \sum_{k=0}^{[r/2]} \frac{(-1)^k F(r-2k-\delta)}{(2k+\delta)!} x^{2k+\delta}. \tag{14}$$

Proof First of all, by setting $z = 2n + 1$ in the functional equation for the Riemann zeta function (see [1])

$$\zeta(1-z) = \frac{2\zeta(z)\Gamma(z)}{(2\pi)^z}\cos\frac{z\pi}{2},$$

we find $\zeta(-2n) = 0,\ n \in \mathbb{N}$. Taking into account the relations of the Riemann zeta functions to Dirichlet functions η and λ (see Lemma 1), we easily notice that they also vanish at even negative integers. Moreover, $\lambda(0) = 0$. It is the other way round with the Catalan function β, *i.e.* it vanishes at odd negative integers. In order to prove this, we need the Hurwitz zeta function $\zeta(z, a)$ initially defined for $\sigma > 1$ ($z = \sigma + i\tau$) by the series

$$\zeta(z, a) = \sum_{k=0}^{+\infty} \frac{1}{(k + a)^z},$$

where a is a fixed real number, $0 < a \leq 1$. The function $\beta(z)$ can now be represented as follows

$$\beta(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k + 1)^z} = 4^{-z} \left(\zeta \left(z, \frac{1}{4} \right) - \zeta \left(z, \frac{3}{4} \right) \right),$$

whence we have

$$\beta(-(2n - 1)) = 4^{2n-1} \left(\zeta \left(-(2n - 1), \frac{1}{4} \right) - \zeta \left(-(2n - 1), \frac{3}{4} \right) \right), \quad n \in \mathbb{N}.$$

Considering that for a non-negative integer n, there holds (see [1]) $\zeta(-n, a) = -B_{n+1}(a)/(n + 1)$, where $B_n(x)$ are Bernoulli polynomials defined by relations

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{n=0}^{+\infty} B_n(x) \frac{t^n}{n!}, \quad B_0(x) = 1,$$

we first replace x with $1 - x$, and have

$$\sum_{n=0}^{+\infty} B_n(1 - x) \frac{t^n}{n!} = \frac{t\mathrm{e}^{(1-x)t}}{\mathrm{e}^t - 1} = \frac{(-t)\mathrm{e}^{(-t)x}}{\mathrm{e}^{-t} - 1} = \sum_{n=0}^{+\infty} B_n(x) \frac{(-1)^n t^n}{n!},$$

whence we obtain $B_n(1 - x) = (-1)^n B_n(x)$ implying $B_{2n}(3/4) = B_{2n}(1/4)$, so that we calculate

$$\beta(-(2n - 1)) = 4^{2n-1} \left(-\frac{B_{2n}(1/4)}{2n} + \frac{B_{2n}(3/4)}{2n} \right) = 0.$$

It is known that for $r \in \mathbb{N}$, the value $\Gamma(1 - r)$ of the gamma function becomes infinite, and we cannot place $\alpha = r$ immediately in Equation (9). In order to get a finite value we take, on the right-hand side sum in Equation (9), the first $m - 1$ terms if $r = 2m - 1$, and the first m terms if $r = 2m$. Then we take limits

$$\lim_{\alpha \to 2m-1} \left(\frac{i}{2} (\mu^{\alpha-1} - (-\mu)^{\alpha-1}) \Gamma(1 - \alpha) - i \sum_{k=0}^{m-1} \frac{\zeta(\alpha - 2k - 1)}{(2k + 1)!} \mu^{2k+1} \right)$$

$$= -i \frac{\log(-\mu) - \log \mu}{2(2m - 2)!} \mu^{2m-2} - i \sum_{k=0}^{m-1} \frac{\zeta(2m - 2k - 2)}{(2k + 1)!} \mu^{2k+1},$$

$$\lim_{\alpha \to 2m} \left(\frac{1}{2} ((-\mu)^{\alpha-1} + \mu^{\alpha-1}) \Gamma(1 - \alpha) + \sum_{k=0}^{m} \frac{\zeta(\alpha - 2k)}{(2k)!} \mu^{2k} \right)$$

$$= -\frac{\log(-\mu) - \log \mu}{2(2m - 1)!} \mu^{2m-1} + \sum_{k=0}^{m} \frac{\zeta(2m - 2k)}{(2k)!} \mu^{2k}.$$

By virtue of $\zeta(-2n) = 0$ $(n \in \mathbb{N})$, the remainders of the series in Equation (9) vanish, so placing $\mu = ix$, $0 < x < 2\pi$, we obtain

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^{2m-1}} = \frac{(-1)^{m-1}\pi x^{2m-2}}{2(2m-2)!} + \sum_{k=0}^{m-1} \frac{(-1)^k \zeta(2m-2k-2)}{(2k+1)!} x^{2k+1},$$
$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m}} = \frac{(-1)^{m}\pi x^{2m-1}}{2(2m-1)!} + \sum_{k=0}^{m} \frac{(-1)^k \zeta(2m-2k)}{(2k)!} x^{2k} \ (m \in \mathbb{N}), \tag{15}$$

which is formula (14) for $s = 1, a = 1, b = 0, c = 1, F = \zeta,\ f = \left\{ {\sin \atop \cos} \right\},\ r = \left\{ {2m-1 \atop 2m} \right\}$.

When dealing with Equation (11), we do not have a problem with the singularity of the gamma function any more, so that we may replace there α with $r \in \mathbb{N}$, and considering that $\eta(-2n) = 0$ $(n \in \mathbb{N})$, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nx}{n^{2m-1}} = \sum_{k=0}^{m-1} \frac{(-1)^k \eta(2m-2k-2)}{(2k+1)!} x^{2k+1},$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos nx}{n^{2m}} = \sum_{k=0}^{m} \frac{(-1)^k \eta(2m-2k)}{(2k)!} x^{2k} \ (m \in \mathbb{N}), \tag{16}$$

which is formula (14) for the choice of parameters $s = -1, a = 1, b = 0, c = 0, F = \eta,\ f = \left\{ {\sin \atop \cos} \right\}\ r = \left\{ {2m-1 \atop 2m} \right\}$.

In the case of Equation (12), there appears again the singularity of $\Gamma(1-r)$ at $r \in \mathbb{N}$, so quite similarly as above we take limits, and after placing $\mu = ix,\ 0 < x < \pi$, because $\lambda(-2n) = 0$ $(n \in \mathbb{N})$, we obtain

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^{2m-1}} = \frac{(-1)^{m-1}\pi x^{2m-2}}{4(2m-2)!} + \sum_{k=0}^{m-1} \frac{(-1)^k \lambda(2m-2k-2)}{(2k+1)!} x^{2k+1},$$
$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^{2m}} = \frac{(-1)^{m}\pi x^{2m-1}}{4(2m-1)!} + \sum_{k=0}^{m} \frac{(-1)^k \lambda(2m-2k)}{(2k)!} x^{2k} \ (m \in \mathbb{N}), \tag{17}$$

which is formula (14) for the choice of $s = 1, a = 2, b = 1, c = 1/2, F = \lambda,\ f = \left\{ {\sin \atop \cos} \right\}\ r = \left\{ {2m-1 \atop 2m} \right\}$.

Finally, in Equation (13), similarly as for Equation (11), we do not have to deal with the singularity of the gamma function and replacing α with $r \in \mathbb{N}$, considering that $\beta(-2n+1) = 0$ $(n \in \mathbb{N})$, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(2n-1)x}{(2n-1)^{2m}} = \sum_{k=0}^{m} \frac{(-1)^k \beta(2m-2k-1)}{(2k+1)!} x^{2k+1},$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(2n-1)x}{(2n-1)^{2m-1}} = \sum_{k=0}^{m-1} \frac{(-1)^k \beta(2m-2k)}{(2k)!} x^{2k} \ (m \in \mathbb{N}), \tag{18}$$

which is formula (14) for the choice of parameters $s = -1, a = 2, b = 1, c = 0, F = \beta,\ f = \left\{ {\sin \atop \cos} \right\} r = \left\{ {2m \atop 2m-1} \right\}$, wherby we complete the proof. ■

We note that because of $\lambda(0) = 0$ the upper bounds in Equation (17) are actually $m-2$ and $m-1$ and $m-1$ for the first sum in Equation (18) since $\beta(-1) = 0$, but purely for the sake of fitting them into a general formula (14), we retain $m-1$ and m in Equation (17), and m for the first sum in Equation (18), without formally changing values.

3.1. *Some applications*

Now we are going to present significant and important applications of our closed form formula (14).

3.1.1. *Integral transforms*

Applying some of the integral transforms, one can obtain various series in closed form. For instance, if we take $f = \cos$, $\delta = 0$ in Equation (14), then apply the Laplace transform, we have

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} p}{(an-b)^r (p^2 + (an-b)^2)} = (-1)^{[r/2]} \frac{c\pi}{2p^r} + \sum_{i=0}^{[r/2]} \frac{(-1)^i F(r-2i)}{p^{2i+1}}.$$

Further, if we set $s = 1$, $a = 1$, $b = 0$, then $F = \zeta$, $c = 1$ must be taken, and we obtain

$$\sum_{n=1}^{\infty} \frac{p}{n^r (p^2 + n^2)} = (-1)^{[r/2]} \frac{\pi}{2p^r} + \sum_{i=0}^{[r/2]} \frac{(-1)^i \zeta(r-2i)}{p^{2i+1}}. \tag{19}$$

Now we apply the inverse Mellin transform to this series, knowing that (see [6, p. 166, 2.16 and p. 167, 2.25])

$$M^{-1}\left(\frac{z}{z^2+n^2}\right) = \begin{cases} \cos(n \log x) & x < 1 \\ 0 & x > 1, \end{cases} \qquad M^{-1}\left(\frac{1}{z^\nu}\right) = \begin{cases} \dfrac{(\log(1/x))^{\nu-1}}{\Gamma(\nu)} & x < 1 \\ 0 & x > 1, \end{cases}$$

coming to the sum of a trigonometric series

$$\sum_{n=1}^{\infty} \frac{\cos(n \log x)}{n^r} = \frac{(-1)^{[r/2]}\pi}{2(r-1)!} \left(\log \frac{1}{x}\right)^{r-1} + \sum_{i=0}^{[r/2]} \frac{(-1)^i \zeta(r-2i)}{(2i)!} \left(\log \frac{1}{x}\right)^{2i}, \quad x < 1.$$

However, if we want to apply the Bessel instead of Mellin transform to series (19), we first refer to (see [5, p. 36, 4.23 and p. 33, 4.6])

$$B\left(\frac{x^{\nu+1/2}}{(a^2+x^2)^\mu}\right) = \frac{a^{\nu-\mu+1}\, y^{\mu-1/2}}{2^{\mu-1}\,\Gamma(\mu)} K_{\nu-\mu+1}(ay)$$

and

$$B(x^\mu) = \frac{2^{\mu+1/2}\,\Gamma((\nu/2)+(\mu/2)+(3/4))}{y^{\mu+1}\,\Gamma((\nu/2)-(\mu/2)+(1/4))},$$

where K_ν is Hankel's function. Thus we obtain the sum of one of the Schlömilch series

$$\sum_{n=1}^{\infty} \frac{K_{1/2}(ny)}{n^{r-(1/2)}} = \frac{(-1)^{[r/2]}\pi y^{r-(3/2)}\Gamma(1-(r/2))}{2^{r+1/2}\Gamma((r+1)/2)} + \sum_{i=0}^{[r/2]} \frac{(-1)^i \zeta(r-2i)\Gamma((1/2)-i)}{i!\,2^{2i+1/2}}\, y^{2i-1/2}.$$

3.1.2. *Convergence acceleration*

On the basis of Equation (14), the convergence acceleration of trigonometric series is obtained. As an example, we consider the series

$$T = \sum_{n=1}^{\infty} \frac{n}{n^2+1} \sin nx. \tag{20}$$

For each natural number M we prove, by the method of mathematical induction, that there holds

$$\begin{aligned} \frac{n}{n^2+1} &= \frac{n}{n^2} \cdot \frac{1}{1+(1/n^2)} = \frac{1}{n}\left(1 - \frac{1}{n^2} + \frac{1}{n^4} - \cdots\right) \\ &= \sum_{m=1}^{M} \frac{(-1)^{m-1}}{n^{2m-1}} + \frac{(-1)^M}{n^{2M-1}(n^2+1)}. \end{aligned} \tag{21}$$

Replacing Equation (21) in Equation (20), we have

$$T = \sum_{m=1}^{M} (-1)^{m-1} T_{2m-1}^{\sin} + \sum_{n=1}^{\infty} \frac{(-1)^M \sin nx}{(n^2+1)n^{2M-1}},$$

where $T_{2m-1}^{\sin}$ denotes first of the closed form formulas (15), *i.e.*

$$T_{2m-1}^{\sin} = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2m-1}} = \frac{(-1)^{m-1}\pi x^{2m-2}}{2(2m-2)!} + \sum_{k=0}^{m-1} \frac{(-1)^k \zeta(2m-2k-2)}{(2k+1)!} x^{2k+1}.$$

So, the greater M, the faster convergence of the remaining series, and accordingly we have the faster convergence of the series T. Here is shown how many terms of the remaining series ought to be taken for the given M and accuracy ε

ε	10^{-1}	10^{-2}	10^{-5}	10^{-8}
$M = 1$	3	8	224	7072
$M = 3$	2	2	6	16
$M = 10$	1	2	2	3

4. Limiting values

THEOREM 3 *For the values of α complementary to those in Table 1, there holds*

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^{\alpha}} = \Phi_{\alpha}(x) + \sum_{k=[(\alpha-1)/2]+1}^{\infty} \frac{(-1)^k F(\alpha-2k-\delta)}{(2k+\delta)!} x^{2k+\delta}, \tag{22}$$

where

$$\Phi_{\alpha}(x) = \frac{(-1)^{[(\alpha-1)/2]} c}{(\alpha-1)!} (\psi(\alpha) + \gamma - \log x)\, x^{\alpha-1} + \sum_{k=1}^{[(\alpha-1)/2]} \frac{(-1)^{[(\alpha-1)/2]-k} F(2k+1)}{(\alpha-2k-1)!} x^{\alpha-2k-1}$$

and if $c = 1$, then $F = \zeta$ or if $c = 1/2$, then $F = \lambda$.

Proof We have seen that if we let α tend to $2m-1$ and to $2m$, respectively, in Equation (9), the right-hand side sums truncate because the zeta function vanish at even negative integers, and as limiting values, we obtain polynomials (see Equation (15)). However, if we let α tend to $2m$ and to $2m-1$, respectively in Equation (9), we find

$$\lim_{\alpha\to 2m}\left(\frac{i}{2}\,(\mu^{\alpha-1}-(-\mu)^{\alpha-1})\Gamma(1-\alpha)-i\sum_{k=0}^{m-1}\frac{\zeta(\alpha-2k-1)}{(2k+1)!}\,\mu^{2k+1}\right)$$
$$=\frac{i}{(2m-1)!}\left(\psi(2m)+\gamma-\frac{\log(-\mu)+\log\mu}{2}\right)\mu^{2m-1}-i\sum_{k=1}^{m-1}\frac{\zeta(2k+1)}{(2m-2k-1)!}\,\mu^{2m-2k-1}$$

$$\lim_{\alpha\to 2m-1}\left(\frac{1}{2}\,((-\mu)^{\alpha-1}+\mu^{\alpha-1})\Gamma(1-\alpha)+\sum_{k=0}^{m-1}\frac{\zeta(\alpha-2k)}{(2k)!}\,\mu^{2k}\right)$$
$$=\frac{1}{(2m-2)!}\left(\psi(2m-1)+\gamma-\frac{\log(-\mu)+\log\mu}{2}\right)\mu^{2m-2}+\sum_{k=1}^{m-1}\frac{\zeta(2k+1)}{(2m-2k-2)!}\,\mu^{2m-2k-2},$$

where ψ is the digamma function. Setting $\mu=ix$, $0<x<2\pi$, we obtain

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2m}}=\frac{(-1)^{m-1}}{(2m-1)!}\,(\psi(2m)+\gamma-\log x)\;x^{2m-1}$$
$$+\sum_{k=1}^{m-1}\frac{(-1)^{m-1-k}\zeta(2k+1)}{(2m-2k-1)!}\,x^{2m-2k-1}+\sum_{k=m}^{\infty}\frac{(-1)^{k}\zeta(2m-2k-1)}{(2k+1)!}\,x^{2k+1},$$
$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2m-1}}=\frac{(-1)^{m-1}}{(2m-2)!}\,(\psi(2m-1)+\gamma-\log x)\;x^{2m-2}$$
$$+\sum_{k=1}^{m-1}\frac{(-1)^{m-1-k}\zeta(2k+1)}{(2m-2k-2)!}\,x^{2m-2k-2}+\sum_{k=m}^{\infty}\frac{(-1)^{k}\zeta(2m-2k-1)}{(2k)!}\,x^{2k},\tag{23}$$

which is Equation (22) for $s=1, a=1, b=0, c=1, F=\zeta, f=\left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}\ \alpha=\left\{\begin{smallmatrix}2m\\ 2m-1\end{smallmatrix}\right\}\ \delta=\left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$.

In the case of Equation (12), bearing in mind $\lambda(z)=(1-2^{-z})\zeta(z)$, and applying the same procedure as above yields

$$\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^{2m}}=\frac{(-1)^{m-1}}{2(2m-1)!}\,(\psi(2m)+\gamma-\log x)\;x^{2m-1}$$
$$+\sum_{k=1}^{m-1}\frac{(-1)^{m-1-k}\lambda(2k+1)}{(2m-2k-1)!}\,x^{2m-2k-1}+\sum_{k=m}^{\infty}\frac{(-1)^{k}\lambda(2m-2k-1)}{(2k+1)!}\,x^{2k+1},$$
$$\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^{2m-1}}=\frac{(-1)^{m-1}}{2(2m-2)!}\,(\psi(2m-1)+\gamma-\log x)\;x^{2m-2}$$
$$+\sum_{k=1}^{m-1}\frac{(-1)^{m-1-k}\lambda(2k+1)}{(2m-2k-2)!}\,x^{2m-2k-2}+\sum_{k=m}^{\infty}\frac{(-1)^{k}\lambda(2m-2k-1)}{(2k)!}\,x^{2k},$$

which is Equation (22) for $s=1, a=2, b=1, c=1/2, F=\lambda, f=\left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}\ \alpha=\left\{\begin{smallmatrix}2m\\ 2m-1\end{smallmatrix}\right\}\ \delta=\left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$. ■

Remark 2 We note that formulas (23) can be obtained by letting in Equation (2) α tend to $2m$ if $f = \sin$ or to $2m - 1$ if $f = \cos$ ($m \in \mathbb{N}$), since we encounter division by zero. Apart from this, the left-hand side series are known as *Clausen functions* defined by (see [4])

$$\mathrm{Cl}_{2\nu}(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2\nu}}, \quad \mathrm{Cl}_{2\nu-1}(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2\nu-1}}, \quad \nu \in \mathbb{N},$$

and, as a by-product, we have managed to express each of them as the sum of a polynomial and a power series in terms of Riemann's ζ function, that is

$$\mathrm{Cl}_{2\nu}(x) = \frac{(-1)^{\nu-1}}{(2\nu-1)!} (\psi(2\nu) + \gamma - \log x)\, x^{2\nu-1} + \sum_{k=1}^{\nu-1} \frac{(-1)^{\nu-1-k}\zeta(2k+1)}{(2\nu-2k-1)!} x^{2\nu-2k-1}$$
$$+ \sum_{k=\nu}^{\infty} \frac{(-1)^k \zeta(2\nu-2k-1)}{(2k+1)!} x^{2k+1}, \quad \nu \in \mathbb{N},$$
$$\mathrm{Cl}_{2\nu-1}(x) = \frac{(-1)^{\nu-1}}{(2\nu-2)!} (\psi(2\nu-1) + \gamma - \log x)\, x^{2\nu-2} + \sum_{k=1}^{\nu-1} \frac{(-1)^{\nu-1-k}\zeta(2k+1)}{(2\nu-2k-2)!} x^{2\nu-2k-2}$$
$$+ \sum_{k=\nu}^{\infty} \frac{(-1)^k \zeta(2\nu-2k-1)}{(2k)!} x^{2k}, \quad \nu \in \mathbb{N}.$$

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