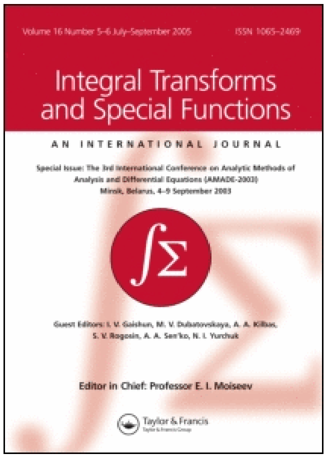

Integral Transforms and Special Functions
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713643686

Series involving the product of a trigonometric integral and a trigonometric function

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Online Publication Date: 01 January 2007
To cite this Article: Tričković, Slobodan B., Vidanović, Mirjana V. and Stanković, Miomir S. (2007) 'Series involving the product of a trigonometric integral and a trigonometric function', Integral Transforms and Special Functions, 18:10, 751 - 763
To link to this article: DOI: 10.1080/10652460701446458
URL: http://dx.doi.org/10.1080/10652460701446458

Integral Transforms and Special Functions
Vol. 18, No. 10, October 2007, 751–763

Taylor & Francis
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Series involving the product of a trigonometric integral and a trigonometric function

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(Received 8 December 2006)

This paper is concerned with the summation of series (1). To find the sum of the series (1) we first derive formulas for the summation of series whose general term contains a product of two trigonometric functions. These series are expressed in terms of Riemann's zeta, Catalan's beta function or Dirichlet functions eta and lambda, and in certain cases, thoroughly investigated here, they can be brought in closed form, meaning that the infinite series are represented by finite sums.

Keywords: Riemann's zeta; Catalan's beta function; Dirichlet; Bessel functions

Mathematics Subject Classification 1991: Primary: 33C10; Secondary: 11M06, 65B10

1. Introduction

In this paper we shall find the sum of the series

$$I_\alpha^{T,f} = \sum_{n=1}^{\infty} \frac{(s)^{n-1} T((an-b)x)}{(an-b)^\alpha} f((an-b)z), \quad \alpha \in \mathbb{R}^+, \tag{1}$$

where $a = \left\{ {1 \atop 2} \right\} b = \left\{ {0 \atop 1} \right\}$, $s = 1$ or -1, f is sin or cos, and T denotes trigonometric integral S_ϕ or C_ϕ defined by

$$S_\phi(x) = \int_0^1 \phi(y) \sin xy \, \mathrm{d}y, \quad C_\phi(x) = \int_0^1 \phi(y) \cos xy \, \mathrm{d}y. \tag{2}$$

To obtain the sum of the series (1) *we do not have to calculate integrals* $T((an-b)x)$ *previously*. At first, we assume that ϕ is integrable. Yet, in order to extend the class of

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Integral Transforms and Special Functions
ISSN 1065-2469 print/ISSN 1476-8291 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/10652460701446458

Table 1.

a	b	s	c	F	for
1	0	1	1	ζ	$0 < x < 2\pi$
		-1	0	η	$-\pi < x < \pi$
2	1	1	1/2	λ	$0 < x < \pi$
		-1	0	β	$-(\pi/2) < x < (\pi/2)$

summable series, we admit that ϕ is differentiable on (0, 1), but not necessarily bounded in the neighborhood of 0 or 1, or not integrable on (0, 1), however, such that there exists at least one of the integrals (2). We further require that the functions $y^k\phi(y)$ $(k \in \mathbb{N})$ are integrable on (0, 1) as well.

Obtaining sums of the series (1) relies on the summation of some trigonometric series. Making use of the method for finding the formula (6) from our article [1] (see Theorem 1, p. 396), we can derive the other particular cases as well, writing all of them, in terms of Riemann's ζ or Catalan's β function or Dirichlet functions η and λ, in the form of a single formula, *i.e.*,

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^{\alpha}} = \frac{c\pi}{2\Gamma(\alpha) f(\pi\alpha/2)} x^{\alpha-1} + \sum_{i=0}^{\infty} \frac{(-1)^i F(\alpha - 2i - \delta)}{(2i+\delta)!} x^{2i+\delta}, \qquad (3)$$

where $\alpha > 0$, $a = \left\{ {1 \atop 2} \right\}$ $b = \left\{ {0 \atop 1} \right\}$, $s = 1$ or -1, and $f = \left\{ {\sin \atop \cos} \right\}$ $\delta = \left\{ {1 \atop 0} \right\}$. The values for F and c are in the table 1, where ζ is Riemann's zeta function $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$, η and λ are Dirichlet functions $\eta(z) = \sum_{k=1}^{\infty}(-1)^{k-1}k^{-z} = (1-2^{1-z})\zeta(z)$, $\lambda(z) = \sum_{k=0}^{\infty}(2k+1)^{-z} = (1-2^{-z})\zeta(z)$, and β is Catalan's function $\beta(z) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-z}$.

Remark 1 We note that the functions ζ, η, λ are analytic in the whole complex plane except for $z = 1$, where they have a pole. The integral representation $\beta(z) = (1/\Gamma(z)) \int_0^{\infty} (x^{z-1}e^x/(e^{2x}+1))\,dx$ of Catalan's function defines an analytical function for Re $z \geq 1$, but also it satisfies the functional equation $\beta(z) = (\pi/2)^{z-1}\Gamma(1-z)\cos(\pi z/2)\beta(1-z)$ extending beta to the left side of the complex plane Re $z < 1$.

Remark 2 After multiplying by $(2(x/2)^{\nu})/(\Gamma(1/2)\Gamma(\nu+(1/2)))$ the integrals (2) with $\phi(y) = (1-y^2)^{\nu-1/2}$, and substituting $y = \cos\theta$, we obtain

$$\varphi_{\nu}(z) = \frac{2(z/2)^{\nu}}{\Gamma(1/2)\Gamma(\nu+(1/2))} \int_0^{\pi/2} \sin^{2\nu}\theta\, g(z\cos\theta)\,d\theta, \qquad (4)$$

where Re $\nu > -(1/2)$, $\varphi_{\nu} = \left\{ {J_{\nu} \atop \mathbf{H}_{\nu}} \right\}$ $g = \left\{ {\cos \atop \sin} \right\}$, J_{ν} and $\mathbf{H}_{\nu}$ are the Bessel and Struve functions respectively of the first kind and order ν. The integrals $S_{\phi}(x)$ and $C_{\phi}(x)$ in equation (2) can be considered as generalizations of Bessel and Struve functions, respectively, since their integral representations (4) are obtained when the particular function $\phi(y) = (1-y^2)^{\nu-1/2}$ is chosen. Just because of this, in this paper we generalize some of our results from [2].

2. Preliminaries

In order to find a sum of the series (1) we set one of the integrals (2) instead of $T(x)$, so that in the sequel we use denotations $T(x) = \left\{ {S_{\phi} \atop C_{\phi}} \right\}$ $g = \left\{ {\sin \atop \cos} \right\}$, meaning that g is one of the trigonometric functions in equation (2), which depends on the choice of S_ϕ or C_ϕ. Afterwards, we shall prove that we are allowed to interchange summation and integration, *i.e.*,

$$\begin{aligned} I_\alpha^{T,f} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z)}{(an-b)^\alpha} \int_0^1 \phi(y) g((an-b)xy)\,\mathrm{d}y \\ &= \int_0^1 \Big(\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z) g((an-b)xy)}{(an-b)^\alpha} \Big) \phi(y)\,\mathrm{d}y \quad (\alpha \in \mathbb{R}^+), \end{aligned} \tag{5}$$

relying on uniform convergence of the right-hand series with respect to y by virtue of Dirichlet's test, which says that (see [3]) the series $\sum_{n=0}^{\infty} a_n(y) b_n(y)$ is uniformly convergent in D, if the partial sums of $\sum_{n=0}^{\infty} a_n(y)$ are uniformly bounded in D, and the sequence $b_n(y)$, being monotonic for every fixed y, uniformly converges to 0. We emphasize that we treat the above right-hand series as a function of $y \in (0, 1)$, regarding x and z as variable parameters.

LEMMA 1 *Let $\phi(y)$ be integrable. Then the right-hand series converges uniformly and there holds equation* (5).

Proof We make use of an elementary trigonometric identity, whereby representing the product of f and g, as the sum of two trigonometric functions sin or cos. Now this series is split up into two series of the type

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} \tau((an-b)(z \pm xy))}{(an-b)^\alpha} \quad (\alpha \in \mathbb{R}^+), \tag{6}$$

where $\tau = \sin$ or $\tau = \cos$. Let us suppose first that $a = 1, b = 0$. If $s = 1$, then there holds

$$\left| \sum_{k=1}^{n} \tau(k(z \pm xy)) \right| \leq \frac{1}{\sin((z \pm xy)/2)} \leq \frac{1}{\sin \varepsilon}$$

for $0 < z \pm xy < 2\pi$, because $\sin((z \pm xy)/2) \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \leq ((z \pm xy)/2) \leq \pi - \varepsilon$.

Similarly, if $s = -1$, we would have

$$\left| \sum_{k=1}^{n} (-1)^{k-1} \tau(k(z \pm xy)) \right| \leq \frac{1}{\cos((z \pm xy)/2)} \leq \frac{1}{\sin \varepsilon}$$

for $-\pi < z \pm xy < \pi$, because $\cos((z \pm xy)/2) \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-(\pi/2) + \varepsilon \leq ((z \pm xy)/2) \leq (\pi/2) - \varepsilon$.

We now suppose $a = 2, b = 1$. If $s = 1$, then

$$\left| \sum_{k=1}^{n} \tau((2k-1)(z \pm xy)) \right| \leq \frac{1}{\sin(z \pm xy)} \leq \frac{1}{\sin \varepsilon}$$

for $0 < z \pm xy < \pi$, because $\sin(z \pm xy) \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \leq z \pm xy \leq \pi - \varepsilon$. Finally, if $s = -1$, then

$$\left| \sum_{k=1}^{n} (-1)^{k-1} \tau((2k-1)(z \pm xy)) \right| \leq \frac{1}{\cos(z \pm xy)} \leq \frac{1}{\sin \varepsilon}$$

for $-(\pi/2) < z \pm xy < (\pi/2)$, because $\cos(z \pm xy) \geq \sin\varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-(\pi/2) + \varepsilon \leq z \pm xy \leq (\pi/2) - \varepsilon$.

So in all these cases, partial sums are uniformly bounded with respect to $y \in (0, 1)$, and on this basis, we determine values of x and z giving rise to this, thus finding boundaries for convergence regions of the series (1) in each of four considered cases. They are in table 2. First, we have immediately $-|x| \leq \pm xy \leq |x|$, and to obtain the condition $0 < z \pm xy < 2\pi$, it is necessary to take $|x| < z < 2\pi - |x|$, from where there follows $|x| < \pi$. In the second case, to satisfy $-\pi < z \pm xy < \pi$, it is necessary to take $|x| - \pi < z < \pi - |x|$, and we have again $|x| < \pi$. In the third, $0 < z \pm xy < \pi$, it is necessary to take $|x| < z < \pi - |x|$, so that we easily find $|x| < (\pi/2)$. Finally, for $-(\pi/2) < z \pm xy < (\pi/2)$, it is necessary to take $|x| - (\pi/2) < z < (\pi/2) - |x|$ giving again $|x| < (\pi/2)$.

On the other hand, the sequence $1/((an-b)^{\alpha})$ obviously monotonically tends to 0 for $\alpha > 0$, and uniformly converges to 0 with respect to y. Together with the above, this proves an uniform convergence of the right-hand series in equation (5) with respect to $y \in (0, 1)$. So the interchange of integration and summation in equation (5) is permitted, and accordingly, there holds equation (5). ∎

LEMMA 2 *Suppose that, for a differentiable on* (0, 1) *and unbounded in the neighborhood of* 0 *or* 1 *function* ϕ, *the integral* $\int_0^1 \phi(y)\mathrm{d}y$ *does not converge. Let there exist at least one of the integrals* $T((an-b)x)$ (T *is* S_ϕ *or* C_ϕ *defined by equation* (2)), *so that* $|T((an-b)x)| \leq M_n(x)$, *and for each corresponding x from table* 2, *the sequence* $(M_n(x))/((an-b)^{\alpha})$, $\alpha > 0$, *monotonically tends to* 0. *Then there holds* (5).

Proof Because of the assumption that ϕ is differentiable function, we know that it is continuous on each closed interval within (0, 1), and, as it is not bounded in the neighborhood of 0 or 1, without loss of generality, we can consider $[\delta, 1-\delta]$, $0 < \delta < 1$. Continuous function on a closed interval is bounded, and referring again to Dirichlet's test, we prove uniform convergence of the right-hand series in equation (5) with respect to y, but this time on $[\delta, 1-\delta]$, so that there holds

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z)}{(an-b)^{\alpha}} \int_{\delta}^{1-\delta} \phi(y) g((an-b)xy)\,\mathrm{d}y$$
$$= \int_{\delta}^{1-\delta} \left(\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z) g((an-b)xy)}{(an-b)^{\alpha}} \right) \phi(y)\,\mathrm{d}y \quad (\alpha \in \mathbb{R}^{+}). \tag{7}$$

We regard the left-hand series as a function of δ with variable parameters z and x. In view of the conditions, by virtue of Dirichlet's test, the left-hand series in equation (7) converges

Table 2.

a	b	s	c	F	Convergence regions
1	0	1	1	ζ	$\{(x,z) : -\pi < x < \pi,\ \lvert x\rvert < z < 2\pi - \lvert x\rvert\}$
1	0	−1	0	η	$\{(x,z) : -\pi < x < \pi,\ \lvert x\rvert - \pi < z < \pi - \lvert x\rvert\}$
2	1	1	1/2	λ	$\{(x,z) : -(\pi/2) < x < (\pi/2),\ \lvert x\rvert < z < \pi - \lvert x\rvert\}$
2	1	−1	0	β	$\{(x,z) : -(\pi/2) < x < (\pi/2),\ \lvert x\rvert - (\pi/2) < z < (\pi/2) - \lvert x\rvert\}$

uniformly with respect to δ on (0, 1). Hence, we have

$$\lim_{\delta\to 0+}\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)z)}{(an-b)^{\alpha}}\int_{\delta}^{1-\delta}\phi(y)g((an-b)xy)\,\mathrm{d}y$$
$$=\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)z)}{(an-b)^{\alpha}}\int_{0}^{1}\phi(y)g((an-b)xy)\,\mathrm{d}y,$$

meaning that the right-hand integral in equation (7) converges, so there holds equation (5). ■

3. Series over the product of trigonometric functions

We have seen that in finding the summation formula for equation (1), the key role plays the series including the product of two trigonometric functions. That is why we are going to investigate them thoroughly. Now, by applying equation (3) to both series in each of the above particular cases, we obtain the following general formula

$$\begin{aligned} S_{\alpha}^{f,g} &= \sum_{n=1}^{\infty}\frac{(s)^{n-1}g((an-b)xy)\,f((an-b)z)}{(an-b)^{\alpha}} \\ &= \frac{c\pi(-1)^{\delta(\delta-d)}}{4\Gamma(\alpha)h(\pi\alpha/2)}\big((z+xy)^{\alpha-1}+(-1)^{\delta}(z-xy)^{\alpha-1}\big) \\ &\quad+\sum_{i=0}^{\infty}\frac{(-1)^{\delta(\delta-d)+i}F(\alpha-2i-d)}{(2i+d)!}\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}(xy)^{2j+\delta}, \end{aligned} \tag{8}$$

where $g=\left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$, and $d=\begin{cases}0 & f=g\\ 1 & f\neq g\end{cases}$, and also there holds $h=\begin{cases}\cos & f=g\\ \sin & f\neq g\end{cases}$. All the other relevant parameters are in table 2.

3.1 *Limiting values of equation (8)*

When on the right-hand side of equation (8) appears $h=\sin$ and $\alpha=2m$ or $h=\cos$ and $\alpha=2m-1$, where $m\in\mathbb{N}$, one should take limit. Let us consider a particular case of the formula (8), taking $a=1$, $b=0$, $s=1$, which implies $c=1$, $F=\zeta$. If $g=\cos$ and $f=\sin$ then $\delta=0$, $h=\sin$, $d=1$, so we have

$$\sum_{n=1}^{\infty}\frac{\sin nz\cos nxy}{n^{\alpha}}=\pi\frac{(z+xy)^{\alpha-1}+(z-xy)^{\alpha-1}}{4\Gamma(\alpha)\sin(\pi\alpha/2)}$$
$$+\sum_{i=0}^{\infty}\frac{(-1)^{i}\zeta(\alpha-2i-1)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j}z^{2i-2j+1}(xy)^{2j}.$$

As we have said, we take limit

$$\lim_{\alpha\to 2m}\Bigg[\pi\frac{(z+xy)^{\alpha-1}+(z-xy)^{\alpha-1}}{4\Gamma(\alpha)\sin(\pi\alpha/2)}$$
$$+\sum_{i=0}^{m-1}\frac{(-1)^{i}\zeta(\alpha-2i-1)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j}z^{2i-2j+1}(xy)^{2j}\Bigg]=G_{2m}(x,y,z),$$

where we have found

$$G_{2m}(x,y,z) = \frac{(-1)^m}{2(2m-1)!}\big((z+xy)^{2m-1}(\log(z+xy)-\psi(2m)-\gamma)$$
$$+ (z-xy)^{2m-1}(\log(z-xy)-\psi(2m)-\gamma)\big)$$
$$+ \sum_{k=1}^{m-1} \frac{(-1)^{k-1}\zeta(2m-2k+1)}{(2k-1)!} \sum_{j=0}^{k-1} \binom{2k-1}{2j} z^{2k-2j-1}(xy)^{2j},$$

where γ is Euler's constant and ψ is the digamma function, $\psi(s) = (\Gamma'(s)/\Gamma(s))$, whose relation to the harmonic numbers $H_n = \sum_{j=1}^{n}(1/j)$ is $\psi(n) = H_{n-1} - \gamma$, with $\psi(1) = -\gamma = \Gamma'(1)$. Finally,

$$\sum_{n=1}^{\infty} \frac{\sin nz \cos nxy}{n^{2m}} = G_{2m}(x,y,z) + \sum_{i=m}^{\infty} \frac{(-1)^i \zeta(2m-2i-1)}{(2i+1)!}$$
$$\times \sum_{j=0}^{i} \binom{2i+1}{2j} z^{2i-2j+1}(xy)^{2j}.$$

Example 1 If we additionally choose $m = 1$, we obtain

$$\sum_{n=1}^{\infty} \frac{\sin nz \cos nxy}{n^2} = z - \frac{1}{2}\left(z\log(z^2-(xy)^2) + xy\log\frac{z+xy}{z-xy}\right)$$
$$+ \sum_{i=1}^{\infty} \frac{(-1)^i\zeta(1-2i)}{(2i+1)!} \sum_{j=0}^{i} \binom{2i+1}{2j} z^{2i-2j+1}(xy)^{2j}.$$

In the same way, we can come to a similar result for any of particular cases if $h = \sin$ and $\alpha = 2m$. Also, we repeat the same procedure, if $h = \cos$ and $\alpha = 2m-1$.

Example 2 If we choose the same parameters as above, but take $f = \cos$ instead of $f = \sin$ and again $m = 1$, then $h = \cos$ and $\alpha = 1$, so we first have to find the limiting value

$$\lim_{\alpha\to 1}\left(\pi\frac{(z+xy)^{\alpha-1}+(z-xy)^{\alpha-1}}{4\Gamma(\alpha)\sin(\pi\alpha/2)} + z\zeta(\alpha)\right) = -\frac{1}{2}\log(z^2-(xy)^2),$$

so that we have

$$\sum_{n=1}^{\infty} \frac{\cos nxy \cos nz}{n} = -\frac{1}{2}\log(z^2-(xy)^2) + \sum_{i=1}^{\infty} \frac{(-1)^i\zeta(1-2i)}{(2i)!} \sum_{j=0}^{i} \binom{2i}{2j} z^{2i-2j}(xy)^{2j}.$$

However, based on the Fourier expansion of the function $-(1/2)\log(2\sin(t/2))$ for $0 < t < 2\pi$, there holds $-(1/2)\log(2\sin(t/2)) = \sum_{n=1}^{\infty}(1/n)\cos nt$, and we can easily find

$$\sum_{n=1}^{\infty} \frac{\cos nxy \cos nz}{n} = -\frac{1}{2}\log\left(4\sin\frac{z-xy}{2}\sin\frac{z+xy}{2}\right),$$

where $0 < z \pm xy < 2\pi$. Relying on this result, we can conclude

$$\sum_{i=1}^{\infty} \frac{(-1)^i\zeta(1-2i)}{(2i)!} \sum_{j=0}^{i} \binom{2i}{2j} z^{2i-2j}(xy)^{2j} = \log\sqrt{\frac{(z-xy)(z+xy)}{4\sin((z-xy)/2)\sin((z+xy)/2)}},$$

which is a new summation formula for the series involving Riemann's zeta function. Further, if we let $xy \to z$, and make use of the combinatorial identity $\sum_{j=0}^{i}\binom{2i}{2j} = 2^{2i-1}$, the above summation formula will be reduced to

$$\sum_{i=1}^{\infty} \frac{(-4z^2)^i \zeta(1-2i)}{(2i)!} = \log\left(\frac{z}{\sin z}\right).$$

3.2 *Closed form cases and some applications of equation (8)*

If $\alpha - d = 2m$ and $F = \zeta, \eta, \lambda$ or $\alpha - d = 2m - 1$ and $F = \beta$ $(m \in \mathbb{N})$, the sum of the series on the right-hand side of equation (8) consists of a finite number of terms because the functions ζ, η, λ vanish at even negative integers, and the function β vanishes at odd negative integers. We shall denote $\alpha = 2m + d - \varepsilon$, where $\varepsilon = \begin{cases} 0, & F = \zeta, \eta, \lambda \\ 1, & F = \beta \end{cases}$. So, for this choice of parameters, the formula (8) is brought into a so-called closed form

$$\begin{aligned} S_{2m+d-\varepsilon}^{f,g} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} g((an-b)xy)\, f((an-b)z)}{(an-b)^{2m+d-\varepsilon}} \\ &= \frac{c\pi}{2} \sum_{j=0}^{m} \binom{2m+d-\varepsilon-1}{2j+\delta} \frac{(-1)^{\delta(\delta-d)} z^{2m-2j+d-\delta-\varepsilon-1} (xy)^{2j+\delta}}{(2m+d-\varepsilon-1)!\, h(m\pi + ((d-\varepsilon)\pi)/2)} \\ &\quad + \sum_{i=0}^{m} \frac{(-1)^{\delta(\delta-d)+i} F(2m-2i-\varepsilon)}{(2i+d)!} \sum_{j=0}^{i} \binom{2i+d}{2j+\delta} z^{2i-2j+d-\delta} (xy)^{2j+\delta}, \end{aligned} \qquad (9)$$

where $g = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$ $\delta = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}$, and independently of that $d = \begin{cases} 0 & f = g \\ 1 & f \neq g \end{cases}$, $h = \begin{cases} \cos & f = g \\ \sin & f \neq g \end{cases}$. The other relevant parameters are in table 2. The formula (9) comprises some particular results from [4], but it is more suitable for immediately obtaining sums of some infinite series as well. For example, in [2], §5.4.15, entry 8, we find

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin nxy \cos nz = \frac{xy}{12}\left(\pi^2 - (xy)^2 - 3z^2\right).$$

It holds if $|xy \pm z| \le \pi$, which is the condition for uniform convergence of the left-hand side series in equation (8) with $a = 1, b = 0$ and $s = -1$. After referring to table 2 and the conditions following equation (9), we obtain the above formula, placing in equation (9): $g = \sin$, $f = \cos$, $m = 1, d = 1, \varepsilon = 0, c = 0$, $F = \eta, \delta = 1, h = \sin$.

3.3 *Some applications*

In [5] the solution of the boundary value problem

$$U_{tt}'' = a^2 U_{xx}'', \quad U(x,0) = \frac{4hx(L-x)}{L^2}, \quad U_t'(x,0) = 0, \ \ 0 \le x \le L, \ \ t \ge 0,$$

is given by the infinite series

$$U(x,t) = \frac{32h}{\pi^3} \sum_{n=1}^{\infty} \frac{\cos((\pi(2n-1)at)/L)\sin(((2n-1)\pi x)/L)}{(2n-1)^3}.$$

However, using equation (9), where we take $m = 1,\ a = 2,\ b = 1,\ s = 1,\ c = (1/2)$ and $F = \lambda$, then $\delta = 0,\ d = 1$ and $\varepsilon = 0$, and denoting $(at\pi/L) = xy$, $(x\pi/L) = z$, we obtain its solution in closed form

$$U(x,t) = \frac{4h}{L^2}(xL - x^2 - a^2t^2),\ \ 0 \le \frac{at}{L} \le \frac{1}{2},\ \ \left|\frac{at}{L}\right| \le \frac{x}{L} \le 1 - \left|\frac{at}{L}\right|.$$

Also (see [5]), the solution of the boundary value problem

$$\begin{aligned} U''_{tt} &= U''_{xx} + x(x-L), \quad U(x,0) = U'_t(x,0) \\ &= U(0,t) = U(L,t) = 0,\ \ 0 \le x \le L,\ t \ge 0, \end{aligned}$$

is

$$U(x,t) = \frac{8L^4}{\pi^5}\sum_{n=1}^{\infty}\frac{\cos(\pi t(2n-1)/L)\sin(\pi x(2n-1)/L)}{(2n-1)^5} - \frac{x}{12}(x^3 - 2x^2L + L^3).$$

Applying equation (9) with $m = 2$, using the same parameters as above, but denoting $(t\pi/L) = xy$ and $(x\pi/L) = z$, this solution in closed form becomes

$$U(x,t) = \frac{1}{2}x^2t^2 - \frac{1}{2}Lxt^2 + \frac{1}{12}t^4, \quad 0 \le t \le \frac{L}{2},\ \ t \le x \le L - t.$$

4. Sum of the series (1)

Now we make use of equation (8), which is, as we have shown, the formula for finding sum of the right-hand side series of equation (5). We develop in equation (8) the binomials $(z \pm xy)^{\alpha-1}$ into binomial series, and after a rearrangement we actually obtain the summation formula of the left-hand side series in equation (5), which is actually the summation formula for the series (1)

$$\begin{aligned} I_\alpha^{T,f} &= \frac{c\pi(-1)^{\delta(\delta-d)}}{4\Gamma(\alpha)h(\pi\alpha/2)}\sum_{k=1}^{\infty}\binom{\alpha-1}{2k-\delta}z^{\alpha-1-2k+\delta}x^{2k-\delta}\int_0^1 y^{2k-\delta}\phi(y)\mathrm{d}y \\ &+ \sum_{i=0}^{\infty}\frac{(-1)^{\delta(\delta-d)+i}F(\alpha-2i-d)}{(2i+d)!}\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}x^{2j+\delta}\int_0^1 y^{2j+\delta}\phi(y)\mathrm{d}y, \end{aligned} \tag{10}$$

where $T(x) = \left\{\begin{smallmatrix} S_\phi(x) \\ C_\phi(x)\end{smallmatrix}\right\}$ $g = \left\{\begin{smallmatrix}\sin \\ \cos\end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix}1 \\ 0\end{smallmatrix}\right\}$, $h = \begin{cases}\cos, & f = g \\ \sin, & f \ne g\end{cases}$ and $d = \begin{cases}0, & f = g \\ 1, & f \ne g\end{cases}$. The other relevant parameters are in table 2.

4.1 *Limiting values*

If $h = \sin$ and $\alpha = 2m$ or $h = \cos$ and $\alpha = 2m - 1$ ($m \in \mathbb{N}$), we encounter a singularity in the first term of equation (10). That is why the limit should be taken. We shall demonstrate it for the following choice of parameters: $a = 1,\ b = 0,\ s = 1,\ c = 1,\ F = \zeta,\ g = \sin,\ f = \cos,$ $\delta = 1,\ h = \sin$ and $d = 1$, and, apart from this, $\phi(y) = y^{-1}$, so that equation (10) becomes

$$I_\alpha^{S_\phi,\cos} = \sum_{n=1}^{\infty} \frac{\cos nz}{n^\alpha} \int_0^1 \frac{\sin nxy}{y}\,\mathrm{d}y = \frac{\pi}{2\Gamma(\alpha)\sin(\pi\alpha/2)} \sum_{k=1}^{\infty} \binom{\alpha-1}{2k-1} \frac{z^{\alpha-2k}x^{2k-1}}{2k-1}$$
$$+ \sum_{i=0}^{\infty} \frac{(-1)^i \zeta(\alpha-2i-1)}{(2i+1)!} \sum_{j=0}^{i} \binom{2i+1}{2j+1} \frac{z^{2i-2j}x^{2j+1}}{2j+1}.$$

Having chosen $\phi(y) = 1/y$, the integral $\int_0^1 (\mathrm{d}y/y)$ does not converge, but there exists the integral $\int_0^1 (\sin nxy/y)\mathrm{d}y = \mathrm{Si}(nx)$, where $\mathrm{Si}(z) = \int_0^z (\sin t/t)\mathrm{d}t$ is the integral sine, which could not be elementarily calculated, but we know it is bounded by $\pi/2$, so Lemma 2 holds. Irrespective of this, without even trying to find its value, and simply by applying equation (10), we obtain the above sum. Yet, because of $h = \sin$, there still remains to take the limit for $\alpha = 2m,\ m \in \mathbb{N}$, *i.e.*,

$$\lim_{\alpha\to 2m} \Bigg[\frac{\pi}{2\Gamma(\alpha)\sin(\pi\alpha/2)} \sum_{k=1}^{m} \binom{\alpha-1}{2k-1} \frac{z^{\alpha-2k}x^{2k-1}}{2k-1}$$
$$+ \sum_{i=0}^{m-1} \frac{(-1)^i \zeta(\alpha-2i-1)}{(2i+1)!} \sum_{j=0}^{i} \binom{2i+1}{2j+1} \frac{z^{2i-2j}x^{2j+1}}{2j+1} \Bigg] = G_{2m}(x,z),$$

and we find

$$G_{2m}(x,z) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}\zeta(2m-2k+1)}{(2k-1)!} \sum_{j=0}^{k-1} \binom{2k-1}{2j+1} \frac{z^{2k-2j-2}x^{2j+1}}{2j+1} + \frac{(-1)^m}{(2m-1)!}$$
$$\times \left(\sum_{k=1}^{m-1} \frac{\log z - \psi(2m-2k+1) - \gamma}{2k-1} \binom{2m-1}{2k-1} x^{2k-1} z^{2m-2k} + \frac{x^{2m-1}\log z}{2m-1} \right)$$

as well as

$$\lim_{\alpha\to 2m} \frac{\pi}{2\Gamma(\alpha)\sin(\pi\alpha/2)} \sum_{k=m+1}^{\infty} \binom{\alpha-1}{2k-1} \frac{z^{\alpha-2k}x^{2k-1}}{2k-1}$$
$$= \frac{(-1)^{m-1}z^{2m-1}}{(2m)!} \sum_{k=m+1}^{\infty} \frac{(x/z)^{2k-1}}{\binom{2k-1}{2m}(2k-1)},$$

so that we get

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^{2m}} \int_0^1 \frac{\sin nxy}{y}\,\mathrm{d}y = G_{2m}(x,z) + \frac{(-1)^{m-1}z^{2m-1}}{(2m)!} \sum_{k=m+1}^{\infty} \frac{(x/z)^{2k-1}}{\binom{2k-1}{2m}(2k-1)}$$
$$+ \sum_{i=m}^{\infty} \frac{(-1)^i \zeta(2m-2i-1)}{(2i+1)!} \sum_{j=0}^{i} \binom{2i+1}{2j+1} \frac{z^{2i-2j}x^{2j+1}}{2j+1}.$$

For instance,

$$\sum_{n=1}^{\infty}\frac{\cos nz}{n^4}\int_0^1\frac{\sin nxy}{y}\,\mathrm{d}y = -\frac{11x^3}{108}-\frac{19\,x\,z^2}{18}+\left(\frac{x^2\,z}{4}+\frac{11z^3}{36}\right)\mathrm{arcth}\frac{x}{z}$$
$$+\frac{x^5}{24z^2}\Phi\Big(\frac{x^2}{z^2},2,\frac{5}{2}\Big)+\left(\frac{x^3}{36}+\frac{x\,z^2}{4}\right)\log\left(1-\frac{x^2}{z^2}\right)$$
$$+\left(\frac{x^3}{18}+\frac{x\,z^2}{2}\right)\log z + x\,\zeta(3)$$
$$+\sum_{i=2}^{\infty}\frac{(-1)^i\zeta(3-2i)}{(2i+1)!}\sum_{j=0}^{i}\binom{2i+1}{2j+1}\frac{z^{2i-2j}x^{2j+1}}{2j+1},$$

where Φ is the Lerch transcendent function defined by the series $\Phi(z,s,\alpha)=\sum_{n=0}^{\infty}(z^n/(n+\alpha)^s)$, and $|x|<\pi$, $|x|<z<2\pi-|x|$ (see table 2).

In the same way, we can come to a similar result for any of particular cases.

4.2 *Closed form cases*

They ensue if $\alpha-d=2m$ and $F=\zeta,\eta,\lambda$ or $\alpha-d=2m-1$ and $F=\beta$ ($m\in\mathbb{N}$). The other relevant parameters and convergence regions are in table 2. When we choose the function ϕ in equation (10), we additionally have to calculate both integrals having in fact a similar structure.

We have already seen that the formula (9) contains all closed form cases for the product of trigonometric functions. So, using the formula (9) we bring the sum of the series (1) into closed form:

$$\begin{aligned} I_{2m+d-\varepsilon}^{T,f} &= \sum_{n=1}^{\infty}\frac{(s)^{n-1}T((an-b)x)}{(an-b)^{2m+d-\varepsilon}}f((an-b)z)\\ &= \frac{c\pi}{2}\sum_{j=0}^{m}\binom{2m+d-\varepsilon-1}{2j+\delta}\frac{(-1)^{\delta(\delta-d)}z^{2m-2j+d-\delta-\varepsilon-1}x^{2j+\delta}}{(2m+d-\varepsilon-1)!\,h(m\pi+((d-\varepsilon)\pi)/2)}\\ &\quad\times\int_0^1 y^{2j+\delta}\phi(y)\mathrm{d}y+\sum_{i=0}^{m}\frac{(-1)^{\delta(\delta-d)+i}F(2m-2i-\varepsilon)}{(2i+d)!}\\ &\quad\times\sum_{j=0}^{i}\binom{2i+d}{2j+\delta}z^{2i-2j+d-\delta}x^{2j+\delta}\int_0^1 y^{2j+\delta}\phi(y)\mathrm{d}y, \end{aligned} \tag{11}$$

where $T(x)=\left\{\begin{smallmatrix}S_\phi(x)\\ C_\phi(x)\end{smallmatrix}\right\}$ $g=\left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$, and independently of that $d=\begin{cases}0 & f=g\\ 1 & f\neq g\end{cases}$, $h=\begin{cases}\cos & f=g\\ \sin & f\neq g\end{cases}$. For $F=\zeta,\eta,\lambda$ there holds $\varepsilon=0$, but for $F=\beta$ it is $\varepsilon=1$. The other relevant parameters are in table 2.

Example 3 First we take $a=1$, $b=0$ and $s=1$ in equation (11), there follows $c=1$ and $F=\zeta$ (see table 2), then $\varepsilon=0$. We choose $\phi(y)=\mathrm{ctg}y$. For $g=\cos$ the integral $\int_0^1\mathrm{ctg}y\cos nxy\,\mathrm{d}y$ does not exist, so there must be $g=\sin$, which means $\delta=1$. If we take

$f = \cos$, then $h = \sin$ and $d = 1$ because $f \neq g$. Let $m = 1$. For $0 < |x| < \pi$, we have

$$\left| x \int_0^1 \operatorname{ctg} y \sin nxy \, \mathrm{d}y \right| = \left| \int_0^x \operatorname{ctg}(t/x) \sin nt \, \mathrm{d}t \right| \leqslant \int_0^x \left| \operatorname{ctg}(t/x) \sin nt \right| \mathrm{d}t$$
$$\leqslant \int_0^\pi \left| \operatorname{ctg}(t/\pi) \sin nt \right| \mathrm{d}t.$$

Also, $|\operatorname{tg}(t/\pi)| > (|t|/\pi)$ implies $|\operatorname{ctg}(t/\pi)| < (\pi/|t|)$, for $|t| < \pi$. Additionally, $\lim_{t\to 0+} \operatorname{ctg}(t/\pi) \sin nt = n\pi$. The function $|\operatorname{ctg}(t/\pi) \sin nt|$ is non-negative, fast oscillatory, and its maximal value between two consecutive zeros constantly decreases. There follows

$$\int_0^\pi \left| \operatorname{ctg}\frac{t}{\pi} \sin nt \right| \mathrm{d}t = \int_0^{1/n} \left| \operatorname{ctg}\frac{t}{\pi} \sin nt \right| \mathrm{d}t + \int_{1/n}^\pi \left| \operatorname{ctg}\frac{t}{\pi} \right| \cdot \left| \sin nt \right| \mathrm{d}t$$
$$< \int_0^{1/n} n\pi \, \mathrm{d}t + \pi \int_{1/n}^\pi \frac{\mathrm{d}t}{t} = \pi(1 + \log n\pi),$$

which means that $\left| \int_0^1 \operatorname{ctg} y \sin nxy \, \mathrm{d}y \right| < (\pi(1 + \log n\pi)/|x|) = M_n(x)$, and we can easily see that for $\alpha > 0$ and each x, $0 < |x| < \pi$, the sequence $(M_n(x)/n^\alpha) \to 0$ monotonically, so that we may apply Lemma 2. Also, we find

$$\int_0^1 y^{2j+1} \operatorname{ctg} y \, \mathrm{d}y = \begin{cases} \log(2\sin 1) + (1/2)\mathrm{Cl}_2(2), & j = 0 \\ \log(2\sin 1) + (3/2)\mathrm{Cl}_2(2) + (3/2)\mathrm{Cl}_3(2) - (3/4)\mathrm{Cl}_4(2), & j = 1 \end{cases}$$

where on the right-hand side are Clausen functions defined by (see [1])

$$\mathrm{Cl}_{2\nu}(x) = \sum_{n=1}^\infty \frac{\sin nx}{n^{2\nu}}, \quad \mathrm{Cl}_{2\nu-1}(x) = \sum_{n=1}^\infty \frac{\cos nx}{n^{2\nu-1}}, \quad \nu \in \mathbb{N}.$$

Making use of equation (9), we obtain

$$I_1^{S_\phi,\cos} = \sum_{n=1}^\infty \frac{\cos nz}{n^3} \int_0^1 \operatorname{ctg} y \sin nxy \, \mathrm{d}y$$
$$= \frac{x}{12}(2\pi^2 - 6\pi z + 3z^2)\left(\log(2\sin 1) + \frac{1}{2}\mathrm{Cl}_2(2)\right)$$
$$+ \frac{x^3}{12}\left(\log(2\sin 1) + \frac{3}{2}\mathrm{Cl}_2(2) + \frac{3}{2}\mathrm{Cl}_3(2) - \frac{3}{4}\mathrm{Cl}_4(2)\right),$$

where $|x| < \pi$ and $|x| < z < 2\pi - |x|$ (see table 2).

Example 4 Let $a = 1, b = 0$ and $s = -1$ in equation (11), implying $c = 0$, $F = \eta$ and $\varepsilon = 0$. For $g = \cos$ there must be $\delta = 0$. Further, we take $f = \cos$, there follows $h = \cos$, $d = 0$ because $f = g$. Let $m = 2$. We choose $\phi(y) = (1 - y^2)^{-1/2}$. It is unbounded about 1, but integrable on (0, 1), thus satisfying conditions of Lemma 1. So applying equation (11), we find

$$I_4^{C_\phi,\cos} = \sum_{n=1}^\infty (-1)^{n-1} \frac{\cos nz}{n^4} \int_0^1 \frac{\cos nxy}{\sqrt{1 - y^2}} \, \mathrm{d}y$$
$$= \frac{7\pi^5}{1440} - \frac{\pi^3}{96}(x^2 + 2z^2) + \frac{\pi}{32}\left(\frac{x^4}{8} + x^2 z^2 + \frac{z^4}{3}\right),$$

where $|x| < \pi$ and $|x| - \pi < z < \pi - |x|$ (see table 2).

Remark 3 We note that this particular result can be obtained in a different way. Namely, substituting $y = \cos\theta$, then multiplying with $2/\pi$, we deal with Bessel functions (see equation (4)), and the above series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} J_0(nx)}{n^4} \cos nz,$$

which is the series over the product of Bessel and trigonometric function, and its sum can be obtained by means of the formula on the page 289 in [2].

Example 5 Further, we take $a = 2$, $b = 1$ and $s = 1$ in equation (11). In table 2, we read $c = (1/2)$ and $F = \lambda$. So $\varepsilon = 0$. If we choose $g = \cos$, $f = \sin$, then we have $\delta = 0$, $d = 1$ and $h = \sin$. Let $m = 1$ and $\phi(y) = \log(\sin y)$. It is unbounded in the neighborhood of 0, however $\int_0^1 \log(\sin y)\,\mathrm{d}y = -\log 2 - (1/2)\mathrm{Cl}_2(2)$, and Lemma 1 holds. Applying equation (11), we obtain

$$\begin{aligned} I_1^{S_\phi,\sin} &= \sum_{n=1}^{\infty} \frac{\sin((2n-1)z)}{(2n-1)^3} \int_0^1 \log(\sin y)\cos((2n-1)xy)\,\mathrm{d}y \\ &= \frac{\pi}{96}\Big(6z(z-\pi)(\mathrm{Cl}_2(2)+\log 4) + x^2(6\mathrm{Cl}_2(2)+6\mathrm{Cl}_3(2)-3\mathrm{Cl}_4(2)+\log 16)\Big), \end{aligned}$$

where $|x| < (\pi/2)$ and $|x| - (\pi/2) < z < (\pi/2) - |x|$ (see table 2).

Example 6 Finally, we take in equation (11): $a = 2$, $b = 1$, $s = -1$, $c = 0$, $F = \beta$, $g = \sin$, $f = \sin$, $d = 0$, $\delta = 1$, $h = \cos$, $\varepsilon = 1$ and $m = 2$, and choose $\phi(y) = \log y$. The integral $\int_0^1 \log y\,\mathrm{d}y$ converges, ϕ meets requirements of Lemma 1, and making use of equation (11), the infinite series is brought in closed form

$$\begin{aligned} I_5^{S_\phi,\sin} &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin((2n-1)z)}{(2n-1)^5} \int_0^1 \log y \sin((2n-1)xy)\,\mathrm{d}y \\ &= \frac{\pi x z}{32}\left(\frac{\pi^2}{4} - \frac{x^2}{12} - \frac{z^2}{3}\right), \end{aligned}$$

where $|x| < (\pi/2)$ and $|x| - (\pi/2) < z < (\pi/2) - |x|$ (see table 2).

This series can take another form. At first, by integrating by parts, we come to the relation $\int_0^1 \log y \sin((2n-1)xy)\,\mathrm{d}y = (1/((2n-1)x)) \int_0^1 ((\cos((2n-1)xy) - 1/y)\,\mathrm{d}y$. Afterwards, we prove $\int_0^p ((1-\mathrm{e}^{-t})/t)\mathrm{d}t - \int_p^\infty (\mathrm{e}^{-t}/t)\,\mathrm{d}t = \log p + \gamma$. On this basis, we find $\mathrm{Ci}(p) + \mathrm{Cin}(p) = \log p + \gamma$, where $\mathrm{Ci}(p) = -\int_p^\infty (\cos t/t)\mathrm{d}t$ is the integral cosine and $\mathrm{Cin}(p) = \int_0^p ((1-\cos t)/t)\mathrm{d}t$ its related function. Hence, there follows

$$\begin{aligned} &\int_0^1 \log y \sin((2n-1)xy)\,\mathrm{d}y \\ &\quad = \frac{1}{(2n-1)x}\big(\mathrm{Ci}((2n-1)x) - \gamma - \log((2n-1)x)\big) \quad (x > 0), \end{aligned}$$

and we obtain the sum of a new series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin((2n-1)z)}{(2n-1)^6 x}\big(\mathrm{Ci}((2n-1)x)-\gamma-\log((2n-1)x)\big)$$
$$=\frac{\pi x z}{32}\left(\frac{\pi^2}{4}-\frac{x^2}{12}-\frac{z^2}{3}\right).$$

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